Analysis of an Asynchronous Optical Buffer

W. Rogiest, K. Laevens, D. Fiems and H. Bruneel
SMACS Research Group, Ghent University

Abstract

We investigate the behaviour of an asynchronous optical buffer by means of a continuous-time queueing model. Through a limit procedure, previously obtained results for a discrete-time queueing model are translated to a continuous-time setting. We also show that the same results can be obtained by a direct analysis using Laplace transforms. Closed-form expressions are obtained for the two important cases of exponentially distributed burst sizes and deterministic burst sizes.

The performance of asynchronous optical buffers shows the same characteristics as that of synchronous optical buffers: a reduction in throughput due to the creation of voids on the outgoing channel and a burst loss probability that is strongly influenced by the choice of fiber delay line granularity. The optimal value of the latter depends on the burst size distribution and the offered load.

1 Introduction

The success of optical networking is beyond doubt. Major cities are connected by dense wavelength division multiplexing (DWDM) links, enabling transmission capacities well beyond the Tbit/s. The nodes, however, still perform data processing in the electronic domain. A growing discrepancy between channel capacity and switching capacity is the drive towards all-optical switching. All-optical packet switching (OPS) could alleviate the problem, by processing both data and control in the optical domain [1]. As mature technology is expected to be still a few years away [2], optical burst switching (OBS) has been proposed as an intermediate solution [3][4][5].

Both solutions equally require contention resolution. When two or more data packets arrive at a network node at the same time, and contend for the same output, external blocking occurs. All packets but one are perceived as supererogatory, and have to be dealt with. Next to the obvious choice of dropping all supererogatory packets, literature [6][7] typically presents three solutions: buffering, deflection routing or wavelength conversion. Optical buffering uses Fiber Delay Lines (FDLs) to delay the light, and is regarded as the most effective [7], but comes with the additional cost of the FDLs.

Analytic results concerning the loss and queueing behaviour of optical buffer systems include [8], studying the influence of granularity. Discrete-time analysis is done in [9] and [10], using probability generating functions (pgf’s). The latter paper investigates queueing characteristics of a synchronous optical buffer. The pgf of the scheduling horizon in a buffer of infinite size, as seen by arrivals, is used to derive several measures of interest, such as the system capacity and the loss probability in a finite system.
This paper extends the results for a synchronous optical buffer to the asynchronous case. Results are applicable to unslotted networks. Compared to a slotted (i.e., synchronous) network, an unslotted network is expected to be technologically more complex due to control issues, as indicated in e.g. [11]. However, an unslotted network could turn out opportune, for reasons of robustness and flexibility.

This paper is structured as follows. In the next section, the mathematical model is described and the assumptions we make are given. In Section 3, we derive results for an infinite buffer in equilibrium. We present two approaches, yielding the same results. Either one can use a limit procedure, starting from results for a synchronous system (results for which are briefly recapitulated), or one can analyze the evolution equation of the asynchronous system directly. In Section 4, heuristics are discussed that allow calculating the burst loss probability in a finite optical buffer, relying on results for the corresponding infinite system. Two special examples of the burst size distributions, the exponential and the deterministic one, are given some more attention in Section 5, leading to closed-form expressions of the performance measures of interest. Using these special cases, the accuracy of the proposed heuristics is compared with simulation results. Finally, conclusions are drawn in Section 6.

2 Model

We study a single outgoing WDM channel, where contention is resolved by means of an FDL buffer. The FDL buffer cannot delay bursts for an arbitrary period of time, but only for multiples of a basic unit $D$, called the granularity of the buffer [8]. For that reason, such buffers are also sometimes called degenerate.

Each burst will remain in the buffer exactly for a time $nD$, for some $n \in \{0, \ldots, N\}$, or will be dropped. Here, $N$ represents the size of the FDL buffer, the maximum achievable delay being $ND$. During the analysis in Section 3, we will assume a buffer of infinite size, $N \to \infty$, so that no loss occurs. Results for a finite system will then be obtained by using a heuristic, as explained in Section 4.

Considering the evolution of buffer contents over time, one can distinguish three important variables. Numbering arriving bursts in the order of their arrival, the first variable, the burst inter arrival time $\tau_k$, captures the time between the $k^{th}$ arrival instant and the next. The second variable is the burst size $B_k$, measuring the time needed for its transmission. The third important variable is the scheduling horizon $H_k$ as observed by the $k^{th}$ arrival. This quantity represents the time between the instant of arrival, and the earliest instant by which the previous burst (and all its predecessors) will have left the system.

The relation between these variables is illustrated in Figure 1. Simple inspection leads to the evolution equation of $H_k$,

$$H_{k+1} = [B_k + D \left\lfloor \frac{H_k}{D} \right\rfloor - \tau_k]^+$$

The expression $\left\lfloor x \right\rfloor$ is the ceil of $x$, i.e., the smallest integer greater than or equal to $x$. The notation $[x]^+$ is standard shorthand for $\max(x, 0)$.

When the $k^{th}$ burst sees a scheduling horizon $H_k$ upon arrival, it will have to be delayed for at least that amount to avoid contention. Since the buffer is degenerate, however, this delay cannot
be realized exactly, the closest match being given by $D_k H_k/D$. Delaying and transmitting this burst pushes the scheduling horizon (just after arrival) to

$$B_k + D_k \left[ \frac{H_k}{D} \right]$$

Taking then into account the burst inter arrival time $\tau_k$, accounting for the possibility that the system becomes empty, one easily obtains equation (1).

To analyze this equation, we need to impose certain restrictions on the distribution of $\tau_k$ and $B_k$. We assume the $\tau_k$ to form a sequence of iid (independent and identically distributed) random variables (rv’s), having a common memoryless distribution. The burst sizes $B_k$ also form a sequence of iid rv’s, and can have a general distribution. (To avoid mathematical complications, however, we will assume that the distribution does not have a heavy tail.)

In the following, both continuous-time and discrete-time models are discussed. In the discrete-time domain, a unit of time $\Delta$ (e.g. 1 µsec) is premised as a slot length, and all time-related quantities are expressed as a multiple of this unit. Events that occur during the same slot are perceived as simultaneous. In continuous time, the smallest time division is infinitely small, as no time unit is premised.

Up to now, the discussion is valid for both CT and DT. In the below, every time both CT and DT variables occur under the same name, we denote DT variables with a primet, e.g. $D'_k$, and leave CT variables unchanged, e.g. $D$. In DT, we will use the probability generating function (pgf) of the probability mass function (pmf), in CT we use the Laplace-Stieltjes transform (LST) of the probability density function (pdf).

For example, for the burst sizes $B_k$, we either have

$$B(z) = E[z^{B_k}] = \sum_{n=1}^{\infty} z^n \Pr[B'_k = n]$$
or

\[ B^*(s) = E[e^{-sB_k}] = \int_0^\infty e^{-sx} b(x) dx \]

As for the inter arrival times, being memoryless, we have a geometric distribution in DT, with mean \(1/p\), and an exponential distribution in CT, with arrival intensity \(\lambda\),

\[ \Pr[\tau_k = n] = p \cdot \bar{\rho}^{n-1}, \ n = 1, 2, ... \]

and

\[ \tau(x) = \lambda e^{-\lambda x}, \ x \geq 0 \]

respectively. (We use the standard notation \(\bar{\rho} = 1 - p\).) The expressions for the corresponding pgf and Laplace transform are

\[ \tau(z) = \frac{pz}{1 - \bar{\rho}z} \]

and

\[ \tau^*(s) = \frac{\lambda}{\lambda + s} \]

respectively.

### 3 Analysis

In what follows, we present two approaches to obtain e.g. the Laplace transform of \(H\) in the continuous-time setting. One consists in taking appropriate limits for the slot size becoming infinitely small, mapping results from the discrete-time setting to the continuous-time setting. The other consists in directly analyzing the evolution equation (1) in continuous time.

All derivations assume the system is stable. On this condition, the distributions of \(H_k\) converge, for \(k \to \infty\), to a unique stochastic equilibrium, independent of the initial system conditions. The pgf and Laplace transform of \(H\) to be obtained are associated with these equilibrium conditions. Stability requires the offered load \(\rho\) to be below some maximum value \(\rho_{\text{max}}\), that is, unlike in conventional queues, typically less than unity, see e.g. [12]. This is also commented upon further on.

Before continuing, we note that two separate non-linear effects can be observed in (1): the operation \([x]^+\) and \([x]\). The former effect, related to the non-negativeness of the buffer content, one could call the queueing effect, i.e., we need to analyze

\[ H = [B + F - \tau]^+ \]  \hspace{1cm} (2)

The latter effect, related to the finite granularity of the FDLs, one could call the FDL effect, and calls for analysis of

\[ F = D \left[ \begin{array}{c} H \\ D \end{array} \right] \]  \hspace{1cm} (3)

Note how it maps values of the rv \(H\) to multiples of \(D\). Below, both effects will first be analyzed separately, and results then combined to yield an overall solution.
3.1 Results for synchronous systems

In [10], both the queueing effect \( [x]^{+} \) and the FDL effect \( [x] \) were studied in a discrete-time setting. The queueing effect yielded the following relation between the pgfs of the variables involved:

\[
H(z) = \frac{p}{z-\bar{p}} B(z) F(z) + K' \frac{z - 1}{z - \bar{p}} \tag{4}
\]

The FDL effect lead to following relation:

\[
F(z) = \sum_{k} \frac{1}{D'_{k}} \frac{z^{D'_{k}} - 1}{z_{k} - 1} H(z \epsilon_{k}) \tag{5}
\]

where the summation index \( k \) runs over \(-D'/2 < k \leq D'/2\), taking on values in \( \mathbb{N} \) only. The symbols

\[
\epsilon_{k} = e^{j 2\pi k / D'}
\]

represent the \( D' \) different complex \( D' \)-th roots of unity. (Note that in [10], the summation ran over \( 0 \leq k < D' \). For our present purposes, however, using \(-D'/2 < k \leq D'/2\), turns out to be more convenient.)

Using the property that \( F(z \epsilon_{k}) = F(z) \), which follows directly from the fact that the random variable \( F \) is always an integer multiple of \( D' \), one can combine (4) and (5) to e.g. obtain, after some additional simplifications,

\[
F(z) = K' \left( \frac{\bar{p}^{D'} - 1}{z^{D'} - \bar{p}^{D'}} \right) \cdot \left( 1 - \sum_{k} \frac{1}{D'_{k}} \frac{z^{D'_{k}} - 1}{z_{k} - 1} p B(z \epsilon_{k}) \right)^{-1} \tag{6}
\]

The constant \( K' \) follows from the normalization condition \( F(z) = 1 \), as

\[
K' = \left( \frac{1}{p} - \mathbb{E}[B'] - \frac{D' - 1}{2} - \sum_{k \neq 0} \frac{1}{\epsilon_{k} - 1} \frac{p}{\epsilon_{k} - \bar{p}} B(z \epsilon_{k}) \right) \cdot \left( \frac{D' \bar{p}^{D'} - 1}{1 - \bar{p}^{D'}} \right)^{-1}
\]

Having determined \( F(z) \), \( H(z) \) then follows readily from (4).

3.2 Limit procedure

Our scope is to derive, for the asynchronous system, \( H^*(s) \), the LST of the equilibrium distribution of the scheduling horizon as seen by arrivals. In this section, we discuss a limit procedure, to retrieve \( H^*(s) \) from \( H(z) \).

To correctly convert results from discrete time to continuous time, one should first observe quantities in the discrete domain. Formally, these are always dimensionless, but in practice, the distinction between time-related quantities and counting-related quantities should be made. The former quantities scale to the dimension time in continuous time, the latter do not scale.
The time unit $\Delta$ disappears from the model, and the limit $\Delta \to 0$ should be taken, represented by substituting $\Delta$ by an infinitesimal amount $dt$. The $Z$-transform scales to the Laplace transform using the substitution $z = e^{-s\Delta}$. Granularity size $D'$, which actually represents $D/\Delta$ in absolute time, corresponds with $D/\Delta$. The average inter-arrival time $1/p$ scales accordingly to $1/(\lambda\Delta)$. Concluding, we obtain the following substitutions:

$$
\begin{align*}
\Delta &\leftrightarrow dt \\
z &\leftrightarrow e^{-s\Delta} \\
D' &\leftrightarrow D/\Delta \\
p &\leftrightarrow \lambda\Delta
\end{align*}
$$

(7)

Applying this limit procedure on the DT solution (4) for the queueing effect yields

$$
H^*(s) = \frac{\lambda}{\lambda - s} F^*(s) B^*(s) - K \frac{s}{\lambda - s}
$$

(8)

In taking the limit, here and in the following, the rules of de l’Hôpital need to be applied frequently, to deal with e.g. indeterminate forms of type $0/0$.

Concerning the FDL effect, equation (5) results in

$$
F^*(s) = \sum_k \frac{1}{D} \frac{1 - e^{-sD}}{s + j 2\pi k/D} H^*(s + j 2\pi k/D)
$$

(9)

where $k$ now runs from $-\infty$ to $+\infty$.

Note that $F^*(s)$ is periodical too, in the sense that

$$
F^*(s) = F^*(s + j 2\pi n/D)
$$

(10)

for any $n \in \mathbb{Z}$. Again, this property allows combining (8) and (9) to yield e.g.

$$
F^*(s) = \left( -K \sum_k \frac{1}{D} \frac{1 - e^{-sD}}{\lambda - (s + j 2\pi k/D)} \right) \cdot \left( 1 - \sum_k \frac{1}{D} \frac{1 - e^{-sD}}{\lambda - t} \left. \frac{\lambda B^*(t)}{\lambda - t} \right|_{t=s+j2\pi k/D} \right)^{-1}
$$

Using the identity

$$
\sum_k \frac{1}{D} \frac{1}{\lambda - (s + j 2\pi k/D)} = -\frac{1}{1 - e^{(\lambda - s)d}}
$$

(11)

see Appendix, we can simplify the expression somewhat further into

$$
F^*(s) = \left( K \frac{1 - e^{-sD}}{1 - e^{(\lambda - s)D}} \right) \cdot \left( 1 - \sum_k \frac{1}{D} \frac{1 - e^{-sD}}{\lambda - t} \left. \frac{\lambda B^*(t)}{\lambda - t} \right|_{t=s+j2\pi k/D} \right)^{-1}
$$

(12)

This is exactly the expression we would have found applying the limit procedure directly to equation (6).
The remaining unknown constant \( K \) can be determined, either by applying the limit procedure once more, or by ensuring normalization of \( F^*(s) \). The final result reads

\[
K = \left( \frac{1}{\lambda} - E[B] - \frac{D}{2} - \sum_{k \neq 0} \frac{1}{D} \frac{\lambda}{t - \lambda} \frac{B^*(t)}{t} \bigg| t = s + j2\pi k/D \right) \cdot \left( -\frac{D}{1 - e^{-\lambda D}} \right)^{-1}
\] (13)

Equations (8), (12) and (13) fully specify \( H^*(s) \).

### 3.3 Direct approach

To consolidate the result of subsection (3.2), we show how they can also be obtained directly.

The complexity of the transform-based solution of the queueing effect, as given in (2), critically depends on the exact form of the LST of \( \tau \). (The same goes, in terms of the pgf of \( \tau \), for the discrete-time case, as discussed in e.g. [13].) For exponentially distributed \( \tau \), the complexity is limited, and results in

\[
H^*(s) = \frac{\lambda}{\lambda - s} B^*(s) F^*(s) - K \frac{s}{\lambda - s}
\]
i.e., the result we obtained via the limit procedure. A direct proof is rather straightforward. Introducing, for convenience, an auxiliary rv \( G = B + F \), with pdf \( g(t) \) \((t > 0)\) and LST \( G^*(s) = B^*(s) F^*(s) \), one has

\[
H^*(s) = \int_0^\infty g(t) dt \int_0^\infty \tau(x) dx e^{-s[t-x]^+}
\]

\[
= \int_0^\infty g(t) dt \int_0^\infty \tau(x) dx e^{-s(t-x)} + \int_0^\infty g(t) dt \int_0^\infty \tau(x) dx \left( e^{-st} - e^{-s(t-x)} \right)
\]

\[
= G^*(s) \tau^*(-s) + \int_0^\infty g(t) dt \int_0^\infty \lambda e^{-\lambda t} dx \left( 1 - e^{-st} e^{sx} \right)
\]

\[
= G^*(s) \frac{\lambda}{\lambda - s} + \int_0^\infty g(t) dt \left( e^{-\lambda t} - e^{-st} \frac{\lambda}{\lambda - s} e^{-(\lambda - s)t} \right)
\]

\[
= G^*(s) \frac{\lambda}{\lambda - s} + G^*(\lambda) \left( 1 - \frac{\lambda}{\lambda - s} \right)
\]

\[
= G^*(s) \frac{\lambda}{\lambda - s} - K \frac{s}{\lambda - s}
\]

The second non-linearity to tackle, is the FDL effect, as stated in (3). The transform-based solution can be obtained by expressing \( F^*(s) \) as

\[
F^*(s) = h(0) + \sum_{k=0}^{+\infty} \int_0^D h(u + kD) e^{-s(k+1)D} du
\]

Rewriting the sum in the right hand side by introducing the comb function, we have

\[
F^*(s) = h(0) + \sum_{k=0}^{+\infty} \int_0^D du h(u + kD) e^{-s(k+1)D} \int_0^D dx e^{-s(u-x)} \sum_{l=-\infty}^{+\infty} \delta((u-x) - lD)
\]
Note that the only Dirac pulse actually having an effect is the one for \( u - x = 0 \), i.e., for \( l = 0 \). Applying identity (18), and rearranging some terms, we can proceed as

\[
F^*(s) = h(0) + \sum_{k=0}^{+\infty} \int_{0^+}^{D} du \, h(u + kD)e^{-s(k+1)D} \int_{0^+}^{D} dx \, e^{-s(u-x)} \sum_{l=-\infty}^{+\infty} \frac{1}{D} e^{-j2\pi l(u-x)/D} \\
= h(0) + e^{-sD} \int_{0^+}^{D} dx \sum_{l=-\infty}^{+\infty} \frac{1}{D} e^{(s+j2\pi l/D)x} \sum_{k=0}^{+\infty} \int_{0^+}^{D} du \, h(u + kD)e^{-s(u+kD)}e^{-j2\pi tu/D} \\
= h(0) + e^{-sD} \sum_{l=-\infty}^{+\infty} \frac{1}{D} e^{(s+j2\pi l/D)D} \sum_{k=0}^{+\infty} \int_{0^+}^{D} du \, h(u + kD)e^{-(s+j2\pi l/D)(u+kD)}
\]

In the last step we used the obvious identity

\[ e^{-j2\pi l/kD} = 1 \]

for any \( l, k \in \mathbb{Z} \), which allowed us to arrive at an expression in terms of \( u + kD \) only in the integral for \( u \). That integration then, combined with the sum over \( k \), amounts to integrating over \((0, \infty)\), i.e.,

\[
F^*(s) = h(0) + e^{-sD} \sum_{l=-\infty}^{+\infty} \frac{1}{D} e^{(s+j2\pi l/D)D} - 1 \sum_{k=0}^{+\infty} \int_{0^+}^{D} du \, h(u + kD)e^{-(s+j2\pi l/D)(u+kD)} \\
= h(0) + \sum_{l=-\infty}^{+\infty} \frac{1}{D} \frac{1 - e^{-sD}}{s + j2\pi l/D} (H^*(s + j2\pi l/D) - h(0))
\]

Using identity (11) once more, we find that the terms involving \( h(0) \) cancel out, yielding

\[
F^*(s) = \sum_{l} \frac{1}{D} \frac{1 - e^{-sD}}{s + j2\pi l/D} H^*(s + j2\pi l/D)
\]
as before.

### 4 Heuristics for the burst loss probability

Results up to now related to an optical buffer of infinite size. In order to obtain the burst loss probability (BLP) in a finite system, i.e., a system with only \( N \) fiber delay lines (realizing delays in the set \( \{0, D, \ldots, ND\} \)) one can rely on heuristics, as explained next.

For conventional queues, fed by a Poisson process of bursts of iid size, a relation exists between (the distributions of) the unfinished work in an infinite system and that in a finite system of, say, capacity \( M \), see e.g. [14]. This relation leads to an expression for the loss ratio \( LR \) in the finite system of the form

\[
LR = \frac{(1 - \rho)}{\rho} \frac{Pr[W_\infty > M]}{1 - Pr[W_\infty > M]}
\]

where \( W_\infty \) denotes the unfinished work in the infinite system (as seen by arrivals).
When dealing with degenerate buffers, one can translate this into a heuristic for the BLP

$$BLP \approx \frac{(1 - \rho_{eq})}{\rho_{eq}} \frac{Pr[H_\infty > ND]}{1 - Pr[H_\infty > ND]}$$  \hspace{1cm} (15)$$

Here, $H_\infty$, the scheduling horizon in an infinite optical buffer, fulfills the role of $W_\infty$, $ND$ is the capacity of the system and $\rho_{eq}$ is the so-called equivalent load, i.e., the load on the system taking into account the overhead created by the voids. (Note that formula (14) assumes only excess unfinished work is lost, i.e., bursts arriving at a nearly full system can still be partially buffered, while in our model, a burst that cannot be delayed sufficiently long due to lack of an appropriate delay line, is dropped entirely.)

We can again rely on results in [10] and the limit procedure to find an expression for the equivalent load in the asynchronous setting. One finds

$$\rho_{eq} = \lambda E[B_{eq}] = \lambda \left( E[B] + \frac{D}{2} + \sum_{k \neq 0} \frac{1}{D} \frac{\lambda}{t} B^*(t) \bigg|_{t=s+j2\pi k/D} \right)$$  \hspace{1cm} (16)$$

As for synchronous systems, the mean equivalent burst size $E[B_{eq}]$ consists of the mean burst size, half the delay line granularity and a term taking into account the finer details of the burst size distribution, through its LST $B^*(t)$. One can show that the system becomes unstable when $\lambda$ is such that $\rho_{eq} = 100\%$.

The tail probabilities $Pr[H_\infty > ND]$ that appear in heuristic (15) can be computed by an (approximate) inversion of the LST $H^*(s)$. It was shown in [10] that for synchronous buffers one has

$$Pr[H_\infty > ND] \approx \frac{cst'}{z_0^{ND'}}$$

under rather mild conditions on the burst size distribution. Here, $z_0$ is the dominant pole of $H(z)$ (and of $F(z)$). It is real, positive and larger than 1. The constant $cst'$ follows from residue theory and is given by

$$cst' = \frac{1}{z_0} \frac{D'}{z_0^{D'} - 1} \left( \lim_{z \to z_0} F(z)(z_0 - z) \right)$$

Applying the limit procedure once more, we find that for asynchronous buffers

$$Pr[H_\infty > ND] \approx \frac{cst}{\gamma^N}$$

where, for convenience, we used

$$\gamma = e^{-s_0 D} = \lim_{\Delta \to 0} z_0^{D'}$$

with $s_0$ the dominant pole of $H^*(s)$ and $F^*(s)$ along the negative real line. A simple bisection algorithm (with possibly an initial search for the appropriate starting interval) suffices to determine $\gamma$ numerically. In some cases, an explicit expression can also be found, see e.g. below.
For small BLP, however, a modified heuristic

\[ BLP \approx (1 - \rho_{eq}) \frac{Pr[H_\infty > ND]}{1 - Pr[H_\infty > ND]} \]  

(i.e., dropping the factor \( \rho_{eq} \) in the denominator) turns out to be more accurate. In the following, we will refer to (15) as "heuristic A" and to (17) as "heuristic B" respectively.

We would like to conclude this section by noting that the same heuristics can also be used to evaluate the BLP in overloaded systems, i.e., when the equivalent load exceeds 100%. Strictly speaking, no equilibrium distribution then exists for e.g. \( H_\infty \). The transform \( H^*(s) \) that is used to approximate \( Pr[H_\infty > ND] \), however, remains a proper function. Formally then, one can still compute the quantities \( Pr[H_\infty > ND] \), the only caveat being that \( \gamma \) is then to be found in the interval \([0, 1)\), i.e. \( s_0 > 0 \). The expression for the constant \( \text{cst} \) remains the same. (When the equivalent load is exactly 100%, \( \gamma = 1 \) and \( s_0 = 0 \). In principle, this leads to somewhat modified expressions, since then the dominant pole has multiplicity two instead of one. Here, we do not pursue this issue further.)

For severely overloaded systems, there is a rather simple, intuitive heuristic. In e.g. conventional queues, when \( \rho \to \infty \), the loss ratio will approximately equal

\[ LR \approx \frac{\rho - 1}{\rho} \]

Indeed, since such system will be busy nearly always, the carried load will be nearly one. The lost load then equals \( \rho - 1 \), leading directly to the above approximation. As \( \rho \to \infty \), the (formal) value for \( Pr[W_\infty > M] \to \infty \), thus the same limit is retrieved in formula (14). Not surprisingly then, for degenerate buffers, heuristic A turns out to be more accurate than heuristic B when \( \rho_{eq} \gg 1 \), as we will illustrate shortly by means of a few numerical examples.

5 Special cases

In this section, we apply the formulas derived above, which are quite generally valid, to two special instances of the burst size distribution: the exponential and deterministic. For these special cases, the infinite sum appearing in e.g. equation (12) or (13), can be removed, yielding closed-form formulas for the LSTs and performance measures derived therefrom.

Results given here were obtained via the limiting procedure. Formulas for the corresponding discrete-time systems (where burst sizes are either geometrically distributed or are deterministic) are given in [15]. Their derivation relied on identities resulting from the partial fraction expansion of appropriately constructed rational functions. At the time of writing, we were unable to verify whether similar identities (involving infinite sums) can be used to simplify e.g. equation (6) for \( F^*(s) \) directly.

Here, we merely state the important formulae only, especially focussing on those needed in the above mentioned heuristics for the BLP.

5.1 Exponentially distributed burst sizes

As was the case for the inter arrival times \( \tau \), exponentially distributed burst sizes can be considered as the limit (for slot sizes going to zero) of geometrically distributed burst sizes. If we
denote the mean burst size by the standard notation $E[B] = \mu^{-1}$, the LST of the burst size distribution then takes on the well-known form

$$B(s) = \frac{\mu}{\mu + s}$$

Expression (12) for $F^*(s)$ simplifies significantly to

$$F^*(s) = \frac{1 - e^{-(s+\mu)D}}{1 - e^{-\mu D}} \cdot \frac{\gamma - 1}{\gamma - e^{-sD}}$$

where

$$\gamma = \frac{\mu + \lambda}{\mu e^{-\mu D} + \lambda e^{+\lambda D}}$$

The constant \textit{cst} appearing in the approximation for the tail distribution becomes

$$\text{cst} = \frac{1 - \gamma e^{-\mu D}}{\gamma (1 - e^{-\mu D})}$$

We further obtain

$$\rho_{eq} = 1 + \frac{\lambda D}{\mu + \lambda} \left( \frac{\lambda}{1 - e^{-\mu D}} + \frac{\mu}{1 - e^{+\lambda D}} \right)$$

A similar expression was found in e.g. [16]. In that paper, the authors derive an expression for the LST of the distribution of the equivalent burst size $B_{eq}$, i.e., the burst size taking into account the voids created due to the degenerate structure of the FDL buffer, see above. The analysis proceeds along somewhat different lines than followed here, but results for e.g. the equivalent load are the same. No explicit expression was derived for the LST of the scheduling horizon, however. Using the LST the obtained for the equivalent burst size distribution in the classical Pollaczek-Khinchin formula for the unfinished work in the M/G/1 system, see e.g. [17], does not yield $H^*(s)$ as given here. This comes as no surprise, given that the equivalent burst size depends on the scheduling horizon as seen by the arriving burst, i.e., we are not dealing with a conventional M/G/1 system here.

Note further that, in this case, it is straightforward to verify that

$$\rho_{eq} = 1 \iff \gamma = 1 \iff s_0 = 0$$

as mentioned above. The condition under which $\rho_{eq} = 1$ fully agrees with the one one would find by taking the appropriate limit of the condition derived in [12] for the synchronous case.

With these formulas at hand, one can easily calculate the BLP via one of the heuristics given above. Some numerical results are shown in Figure 2. It compares results from simulation (dots) with those obtained via heuristic A (gray curves) or heuristic B (black curves). The mean burst size was set to 50 $\mu$s (corresponding with about 60 kbytes at 10 Gbps). The granularity $D$ varied from 0 to 100 $\mu$s. The left pane shows results for different values of $N$, the number of available FDLs, under a constant load of $\rho = 60\%$. The right pane, where $N = 20$, shows results for different input load levels.
For small values of $N$, the heuristics are a bit pessimistic, i.e., they overestimate the BLP. Heuristic B is more accurate for low values of the BLP, but does not converge to the right asymptotic value when $\rho_{eq} \gg 1$, as predicted. (Here, $\rho_{eq} \to \infty$ as $D \to \infty$).

There is an optimal granularity $D$, function of the load, as was the case in synchronous systems, see [10]. As we will illustrate next, the optimal value also depends on the burst size distribution.

5.2 Deterministic burst sizes

In this case, all burst are of length $B$. In order to proceed, we need to express $B$ as

$$B = aD - b$$

where $a \geq 1$ and $0 \leq b < d$. That is,

$$a = \left\lceil \frac{B}{D} \right\rceil$$

and

$$b = aD - B$$

With this convention, the limit procedure yields

$$F^a(s) = \frac{a \left(1 - e^{-\lambda D} \right) - e^{-\lambda(D-b)} \left(1 - e^{-sD} \right)}{e^{-saD}e^{-\lambda(D-b)} \left(1 - e^{-sD} \right) + (e^{-\lambda D} - e^{-sD}) \left(1 - e^{-saD} \right)}$$

The equivalent load is now given by

$$\rho_{eq} = 1 + \lambda D \left( a - \frac{e^{-\lambda(D-b)}}{1 - e^{-\lambda D}} \right)$$
and reaches 100% when
\[ a = \frac{e^{-\lambda(D-b)}}{1 - e^{-\lambda D}} \]
again in agreement with what one would obtain by taking the appropriate limit of the condition given in [12] for this specific case.

The dominant pole \( \gamma = e^{-s_0 D} \) now has to be determined as the solution of
\[ \gamma^a e^{-\lambda(D-b)}(1 - \gamma) + (e^{-\lambda D} - \gamma)(1 - \gamma^a) = 0 \]
For \( \rho_{eq} < 100\% \), it is the smallest solution in \((1, \infty)\), for \( \rho_{eq} > 100\% \), it is the only solution in \((0, 1)\).

Finally, the constant needed in the approximation of \( \Pr[H_\infty > ND] \) is given by
\[ \text{cst} = \frac{a \left( 1 - e^{-\lambda D} \right) - e^{-\lambda(D-b)}}{a \left( \gamma - e^{-\lambda D} \right) - \gamma(1 - \gamma^a) - \gamma^a + 1 - e^{-\lambda(D-b)}} \]

Figure 3: BLP for deterministic burst sizes

Figure 3 shows the same set of curves as Figure 2. Again, the (mean) burst size was set to 50 \( \mu \text{sec} \) and the granularity \( D \) varied from 0 to 100 \( \mu \text{sec} \). The shape of the curves is substantially different from the ones in Figure 2, and the BLP can be an order of magnitude smaller. Again, heuristic B is more accurate for lower values of the BLP, but does not converge to the correct limit for \( \rho_{eq} \gg 1 \). There are now several "local optima", when \( B \) is a multiple of \( D \), i.e., for \( b = 0 \). The global optimum value of \( D \) is again function of the load, but not in a continuous fashion, as was the case for exponentially distributed burst sizes.

6 Conclusions

Expressions for various performance measures for asynchronous optical buffers were derived by taking the limit of corresponding results obtained elsewhere for synchronous ones. An analysis
directly in continuous time seems feasible too, but appears to be slightly more complex than the limit procedure (at least to the authors’ opinion). Based on (approximate inversion of) the LST of the scheduling horizon in an infinite system, heuristics were developed to determine the BLP in finite systems. Two special cases of burst size distributions were used to establish the accuracy of these heuristics. For these special cases, the resulting formulas turned out to be relatively simple, allowing for easy numerical evaluation. The examples given further revealed that the BLP is rather sensitive to the choice of the granularity D, as was the case for synchronous optical buffers. The optimal value depends not only on the burst size distribution, but also on the offered load.

**Appendix: proof of identity (11)**

The identity results from a limit procedure too, a more direct proof is based on the well-known Fourier expansion of the comb function

\[
\sum_{k=-\infty}^{+\infty} \delta(x - kD) = \sum_{k=-\infty}^{+\infty} \frac{1}{D} e^{j2\pi kx/D}
\]  

(18)

Using \( u \) to denote \(-(\lambda - s)\), one can cast identity (11) into the form

\[
\sum_{k=-\infty}^{+\infty} \frac{1}{D} \frac{e^{uD} - 1}{u + j2\pi k/D} = e^{uD}
\]  

(19)

The RHS can be written as

\[
e^{uD} = \int_{0^+}^{D} e^{ux} \sum_{k=-\infty}^{+\infty} \delta(x - kD) \, dx
\]

(Since \( 0 < x \leq D \), \( \delta(x - kD) \) contributes to the integration only for \( k = 1 \).) Using (18) and exchanging integration and summation, gives for the RHS

\[
\sum_{k=-\infty}^{+\infty} \int_{0^+}^{D} \frac{1}{D} e^{(u+j2\pi k/D)x} \, dx
\]

from which (19) then follows.

**References**


