Transmitter Buffer Behavior of the Stop-and-Wait ARQ Scheme Under Correlated Errors

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Abstract
This paper presents an analytical approach for analysing the queue length and the packet delay in the transmitter buffer of a system working under a Stop-and-Wait retransmission protocol. The buffer at the transmitter side is modelled as a discrete-time queue with an infinite storage capacity. The numbers of information packets entering the buffer during consecutive slots are assumed to be independent and identically distributed (i.i.d.) random variables. The packets are sent through an unreliable and non-stationary channel, which is modelled by a two-state Markov chain. An explicit formula is derived for the probability generating function (pgf) of the buffer contents. This pgf can then be used to derive several queue-length characteristics as well as the mean packet delay. By means of some numerical examples the effect of error correlation on the system performance is illustrated. Finally, the obtained analytical results are compared with appropriate simulations according to the described model.

1. INTRODUCTION
Automatic repeat request (ARQ) protocols are widely used to ensure reliability for transmitted data. Nowadays, as the demand for a wireless transmission is getting larger, they are even of greater importance than ever. Obviously, the wireless channel is much noisier than a classical wired channel and because of this fact, the errors occurring during data transmissions take place more often. Using ARQ protocols, one may control these errors so that the data delivered to the end user is error free. The working idea of this kind of protocol is as follows. The transmitter sends a packet (code word) consisting of information bits and error detection code. The receiver checks the transmitted packet by a simple decoding technique. If no error is encountered, a positive acknowledgement (ACK) is sent to the transmitter via the feedback channel and the packet is delivered to the end user. Otherwise, if an error is detected, the negative acknowledgement (NACK) is sent back to the transmitter and the erroneous packet is discarded. The transmitter, upon the receipt of the NACK message, simply retransmits the packet. The time, which has elapsed from the transmission of the packet till the moment when the corresponding feedback message arrives, is called the round trip time (RTT) and is expressed in slots (RTT is equal to $s+1$ slots). One slot is the fraction of time, which is necessary to transmit exactly one packet. Generally, there are three basic ARQ schemes: Stop-and-Wait (SW), Go-back-N (GBN) and Selective Repeat (SR).

In this paper the SW protocol is analysed. As opposed to the two other protocols, where the data are transmitted continuously, in the SW protocol the transmitter has to wait for acknowledgement messages from the receiver (before transmitting a next packet) (see Fig. 1). In this case, one can talk about wasted capacity of the channel. Despite the fact that it has a relatively low throughput, the analysis of the SW protocol under the assumption of correlated errors gives us a good physical insight into more sophisticated ARQ systems as well. Furthermore, the implementation of a system using the SW protocol is less complicated. The SR protocol for instance, although the most efficient one, requires a resequencing buffer at the receiver side, causing the whole system to be more complex. Taking all these things into account and the nature of the wireless environment, which is used to transmit data, the SW protocol seems to be a correct choice to analyse the influence of error correlation on ARQ performance. In this study, we mainly focus on the analysis of the queue length in the transmitter buffer. Specifically, we devise an analytical technique to obtain an explicit expression for the probability generating function (pgf) of the buffer contents. From this pgf, several performance characteristics, such as the mean queue length, the tail distribution of the queue length and the mean packet

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delay can be derived.

The outline of the paper is as follows. The system under study and the modelling assumptions are described in Section 2. The details of the mathematical model of the system are explained in Section 3. Section 4 deals with the queuing analysis of the system. In Section 5, some numerical examples are presented and the effect of error correlation on the system performance is discussed. Finally, in Section 6, the conclusions are given.

2. SYSTEM DESCRIPTION

We model the transmitter buffer of a system working under a Stop-and-Wait retransmission protocol as a discrete-time queue with an infinite storage capacity. The numbers of packets entering the buffer during consecutive slots are assumed to be independent and identically distributed (i.i.d.) random variables with common pgf $A(z)$. The information in the form of packets is sent to a receiver over a wireless channel. A packet is kept in the buffer as long as the information about its successful delivery (ACK message) is not received by the transmitter. The feedback message that arrives at the transmitter at the end of slot $k$ is received for a packet that was transmitted during slot $k - s$. The parameter $s$ is called the feedback delay here. The acknowledgement messages are assumed to be always error free. This assumption is justified by the fact that ACK/NACK messages are much shorter than the information packets and are very well protected once coded. The time-varying nature of the wireless channel is modelled by a two-state Markov chain (see Fig. 2), often called Gilbert-Elliott model [6, 7]. The states are termed GOOD (state 0) and BAD (state 1) and evidently model two different conditions of the channel. The probability that the transmitted packet is received with error in a GOOD and BAD state is $e_0$ and $e_1$ respectively, where generally $e_0 < e_1$, although this restriction is not a requirement for the analysis. The GOOD state represents a relatively satisfactory condition of the wireless channel (where the error free transmission of a packet is highly probable), whereas in the BAD state, the proper transmission of a packet is more difficult. From one slot to the next, the channel state changes with probabilities $1 - \alpha$ and $1 - \beta$ respectively, as indicated in Fig. 2. The parameters $\alpha$ and $\beta$ could then be used as the error correlation indicators. However, instead of $\alpha$ and $\beta$ let us use the more intuitive parameters $\sigma$ and $K$, defined as:

$$\sigma \triangleq \frac{1 - \alpha}{2 - \alpha - \beta}, \quad K \triangleq \frac{1}{2 - \alpha - \beta}$$

(1)

The parameter $\sigma$ ($0 < \sigma < 1$) corresponds to the overall fraction of BAD slots, whereas $K$ is a measure for the amount of correlation in the channel. Indeed, the mean sojourn times in states 0 and 1 are given by $K/\sigma$ and $K/(1 - \sigma)$ respectively as opposed to $1/\sigma$ and $1/(1 - \sigma)$ for an uncorrelated channel ($K = 1$). Furthermore, the factor $K$ could be called the correlation factor as the correlation coefficient $\gamma$ between the channel states in two consecutive slots is determined as a simple function of $K$:

$$\gamma = \alpha + \beta - 1 = 1 - \frac{1}{K}$$

(2)

Note that for positive channel correlation, we have $0 < \gamma < 1$ and $K > 1$.

3. MATHEMATICAL MODEL AND SYSTEM EQUATIONS

In order to study the behaviour of the transmitter buffer, let us define first the random variable $u_k$ as the “buffer contents” (i.e. the number of packets in the transmitter buffer) at the beginning of slot $k$. We introduce the random variable $m_k$ as the remaining number of slots needed to receive an acknowledgement message for the packet currently under transmission at the beginning of slot $k$, if $m_k \geq 1$, and $m_k = 0$ if $u_k = 0$. In the sequel, the variable $m_k$ will be referred to as the “residual service time” at the beginning of slot $k$. Note that when $m_k = 1$, it means that the acknowledgement message (ACK or NACK) arrives at the transmitter at the end of slot $k$, and the packet either leaves the buffer or is retransmitted (see Fig. 3). Also, we add a third variable $r_k$, which represents the channel state (0 or 1) during slot $k - s$. Then, the triple
Figure 3: Residual service time.

\[(r_k, m_k, u_k)\] constitutes a three-dimensional Markovian state description of the system at the beginning of slot \(k\). The behaviour of the system can be described by the following system equations, where \(a_k\) denotes the number of packets entering the system during slot \(k\):

(a) If \(m_k = 0\) (and hence \(u_k = 0\)):

\[
u_{k+1} = a_k; \quad m_{k+1} = \begin{cases} 0, & \text{if } a_k = 0, \\ s + 1, & \text{if } a_k > 0. \end{cases}
\]

That is, the packets in the system at the beginning of slot \(k + 1\) are all those which entered the system during slot \(k\). In case of any arrivals to the system during slot \(k\), a packet will be transmitted during the next slot \(k + 1\) and the residual service time becomes \(s + 1\) slots; otherwise the system remains empty.

(b) If \(m_k = 1\) (and hence \(u_k > 0\)):

\[
u_{k+1} = u_k - v_k + a_k; \quad m_{k+1} = \begin{cases} 0, & \text{if } v_k = u_k = 1 \text{ and } a_k = 0; \\ s + 1, & \text{otherwise}. \end{cases}
\]

Here the random variable \(v_k\) represents the type of acknowledgement that is received at the end of slot \(k\), i.e., \(v_k = 1\) in case of an ACK and \(v_k = 0\) in case of a NACK. The distribution of \(v_k\) is completely determined by the value of \(r_k\), i.e., if in slot \(k - s\) the channel was in state \(r_k = i (i = 0, 1)\), the probability of receiving a NACK (ACK) for the transmitted packet at the end of slot \(k\) is \(\text{Prob}[v_k = 0 | r_k = i] = e_i\) and \(\text{Prob}[v_k = 1 | r_k = i] = 1 - e_i\), respectively. When a positive acknowledgement is received (\(v_k = 1\)), the packet leaves the system at the end of slot \(k\); otherwise, the packet must be retransmitted and stays in the system. The system becomes empty when in case of an ACK the packet that leaves is the last one in the system and no new packets arrive to the system in slot \(k\); otherwise the transmission of a new packet (in case of an ACK is received) or a retransmission (in case of a NACK) starts.

(c) If \(m_k > 1\) (\(m_k = 2, \ldots, s + 1\)) and hence \(u_k > 0\):

\[
u_{k+1} = u_k + a_k; \quad m_{k+1} = m_k - 1,
\]

i.e., the packet does not leave the system at the end of slot \(k\), so the evolution of the system occupancy is only due to the new arrivals occurring during slot \(k\); the residual service time is simply decreased by one in this case.

4. STEADY-STATE QUEUEING ANALYSIS

4.1 Generating function of the buffer contents

In this section, we will derive an expression for the pgf of the buffer contents at the beginning of an arbitrary slot in the steady state.

Naturally, the steady state exists only when the average number of packets entering the system is strictly less than the throughput \(\eta\) of the Stop-and-Wait ARQ protocol, i.e., \(A'(t) = E[a_k] < \eta\), where the prime symbol denotes the first derivative of \(A(z)\) with respect to \(z\), so the operator \(E[\ldots]\) stands for the expected value of the expression between the brackets. Note that an expression for the throughput \(\eta\) of the analysed system is different from the classical one (when a static channel is considered). As the time varying nature of the transmission channel is taken into account, the expression for the throughput takes the following form \([9]\):}

\[
\eta = (1 - \sigma) \frac{1 - e_0}{1 + s} + \sigma \frac{1 - e_1}{1 + s}.
\]

Our first step is to define the joint pgf \(P_k(x, y, z)\) of the system state vector \((r_k, m_k, u_k)\) as

\[
P_k(x, y, z) \triangleq \sum_{i=0}^{s+1} \sum_{j=0}^{s+1} \sum_{n=0}^{\infty} \text{Prob}[r_k = i, m_k = j, u_k = n] x^i y^j z^n.
\]

Next, we derive an expression for the function \(P_{k+1}(x, y, z)\), the joint pgf at the beginning of slot \(k + 1\). Based on the system equations (3)-(8), the pgf \(P_{k+1}(x, y, z)\) is calculated as follows:

\[
P_{k+1}(x, y, z) = P_k(x, y, z) \left(1 + \sigma \frac{1 - e_1}{1 + s} \right) x^s y^s z^s.
\]
\[ + \sum_{i=0}^{\infty} \left[ A(0) + (A(z) - A(0))y^{-1}\right] T_i(x) \cdot \frac{1}{y} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \frac{A(0)P[i, 1, l]}{y^{-1+l}} T_l(x) e_i = 1 \]

where we have introduced the shorthand notations \( \text{Prob}[i, j, n] = \text{Prob}[r_k = i, m_k = j, u_k = n] \) and \( \bar{q} = 1 - q \).

Moving ahead with the calculations, we get

\[
P_{k+1}(x, y, z) = \sum_{i=0}^{\infty} \left\{ [A(0) + (A(z) - A(0))y^{-1}] \text{Prob}[i, 0, 0] \cdot T_i(x) + A(0) \text{Prob}[i, 1, 1] \cdot T_1(x) e_i + y^{s+1} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{A(0)P[i, 1, l]}{y^{-1+l}} T_l(x) e_i \right\} = 1.
\]

(15)

The remainder of the analysis is based on the partial pgfs \( R_i(z) \) and \( U_i(y, z) \) which are defined as

\[
R_i(z) = \sum_{n=1}^{\infty} \text{Prob}[i, 1, n] z^{n-1};
\]

\[
U_i(y, z) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \text{Prob}[i, j, n] y^j z^n.
\]

(16) \quad (17)

In the steady state \( (k \to \infty) \), the functions \( P_k(x, y, z) \), \( R_i(z) \) and \( U_i(y, z) \) become independent of \( k \) and converge to some limiting functions \( P(x, y, z) \), \( R_i(z) \) and \( U_i(y, z) \). By taking the limit \( k \to \infty \) in equation (15) and after some further simplifications, we then get the following expression for the joint pgf \( P(x, y, z) \) in terms of the partial pgfs \( U_i(y, z) \) and \( R_i(z) \):

\[
P(x, y, z) = \sum_{i=0}^{\infty} T_i(x) \left\{ A(0)R_i(z)(1 - y^{s+1})\bar{e}_i + p_i \left[ A(0) + (A(z) - A(0))y^{s+1} - \frac{A(z)}{y} \right] \right\} + \frac{1}{y} A(z) U_i(y, z),
\]

(18)

where \( p_i = \lim_{k \to \infty} \text{Prob}[i, 0, 0] \). It is clear that \( p_0 \) is the probability of having an empty buffer. From definitions (10) and (17), it follows that \( P(x, y, z) \) can also be expressed as

\[
P(x, y, z) = U_0(y, z) + U_1(y, z) x.
\]

(19)

Identification of the coefficients of equal powers of \( x \) on the right-hand sides of equations (18) and (19) then yields a set of two linear equations for \( U_0(y, z) \) and \( U_1(y, z) \), which is easily solved as

\[
U_0(y, z)[y - A(z)] = \left[ y(p_0 + p_1\beta) - \gamma A(z) \right] p_1, \quad U_1(y, z)[y - A(z)] = \left[ y(p_0 + p_1\beta) - \gamma A(z) \right] p_1.
\]

(20)

Here the notations \( G \), \( H \) and \( \phi_1(z) \) are defined as

\[
G = [\alpha \bar{e}_0 R_0(0) + \beta \bar{e}_1 R_1(0)];
\]

(22)
\[ H = [\tilde{\sigma}R_0(0) + \beta \tilde{r}_1 R_1(0)]; \]  
\[ \phi(z) = \epsilon_1 z + (1 - \epsilon_1). \]

The next step is of course to determine the two unknown functions \( R_0(z) \) and \( R_1(z) \) and the four unknown parameters \( p_0, R_0(0), p_1 \) and \( R_1(0) \) in order to obtain the complete expression for the pgf \( P(x, y, z) \). From the system equations (3)-(8), we easily observe that \( m_k = 0 \) if and only if \( u_k = 0 \). This observation gives rise to the following two equalities:
\[ U_0(y, 0) = U_0(0, 0) = p_0, \quad \text{for all } y; \]
\[ U_1(y, 0) = U_1(0, 0) = p_1, \quad \text{for all } y. \]

If we now set \( z = 0 \) in the equations (20)-(21) and invoke the above equalities, we find the following relationships between \( p_0, p_1, R_0(0) \) and \( R_1(0) \):
\[ p_0 + p_1 = \frac{A(0)}{1 - A(0)} \left[ \tilde{r}_0 R_0(0) + \tilde{r}_1 R_1(0) \right]; \]
\[ \beta p_1 - \delta p_0 = \frac{\gamma A(0)}{1 - \gamma A(0)} \left[ \tilde{r}_1 \beta_1 R_1(0) - \tilde{r}_0 \sigma R_0(0) \right]. \]

The second step towards the determination of the unknowns from equations (20) and (21) is another very useful remark concerning the partial pgfs \( U_0(y, z) \) and \( U_1(y, z) \). It is clear namely that these functions should be bounded for all values of \( y \) and \( z \) such that \( |y| \leq 1 \) and \( |z| \leq 1 \). Naturally this is also true for \( y_1 \overset{\Delta}{=} A(z) \) or \( y_2 \overset{\Delta}{=} \gamma A(z) \) and \( |z| \leq 1 \), since \( |A(z)| \leq 1 \) and \( |\gamma A(z)| \leq 1 \) for all \( z \) with \( |z| \leq 1 \), since \( A(z) \) is a pgf. If we choose \( y = y_1 \) or \( y = y_2 \) in the set of equations (20) and (21), where \( |z| \leq 1 \), the left-hand sides of these equations vanish. Evidently, the right-hand sides of (20) and (21) also have to be equal to zero. This leads to a set of equations for the functions \( R_0(z) \) and \( R_1(z) \), from which these functions are finally obtained as
\[ R_0(z) = \frac{A(z)^\gamma}{\tilde{\sigma} \Omega_0_{\cdot}(z) \Omega_0(z) + \sigma \Omega_{\cdot\cdot}(z) \Omega_1(z)} \times \left[ \sigma \Omega_{\cdot\cdot}(z)[1 - A(z)](p_0 + p_1) \right. \]
\[ + \Omega_1(z)^\gamma[1 - \gamma A(z)](\sigma p_0 - \sigma p_1)], \]
\[ R_1(z) = \frac{A(z)^\gamma}{\tilde{\sigma} \Omega_0_{\cdot}(z) \Omega_0(z) + \sigma \Omega_{\cdot\cdot}(z) \Omega_1(z)} \times \left[ \sigma \Omega_{\cdot\cdot}(z)[1 - A(z)](p_0 + p_1) \right. \]
\[ - \Omega_0(z)^\gamma[1 - \gamma A(z)](\sigma p_0 - \sigma p_1)]. \]

Here we have introduced the following notations:
\[ \Omega_0(z) = A(z)^{s+1}\phi(z) - z; \]
\[ \Omega_{\cdot}(z) = A(z)^{s+1}\phi_1(z) - z. \]

Note that to derive the equations (29) and (30) the equations (27) and (28) have also been used.

Now, as one can observe, the only remaining unknowns in the equations for \( R_0(z) \) and \( R_1(z) \) (and also in the equations for \( U_0(y, z) \) and \( U_1(y, z) \)) are the probabilities \( p_0 \) and \( p_1 \). One relationship between the two unknowns is simply found from the normalisation condition for \( P(x, y, z) \):
\[ P(1, 1, 1) = U_0(1, 1) + U_1(1, 1) = 1. \]

This results in
\[ A'(1) = R_0(1)\tilde{\alpha}_0 + R_1(1)\tilde{\alpha}_1, \]
where \( A'(1) \) is the mean arrival rate. After some lengthy manipulations, involving multiple applications of de l’Hôpital’s rule, the following, more explicit form of (34) can be found:
\[ (1 - \gamma^{s+1}) \left[ (s+1)A'(1) + (p_0 + p_1 - 1)(1 - \sigma p_0 - \sigma p_1) \right] \]
\[ = \gamma^s(s + 1)(\sigma_0 - \epsilon_1)(\sigma p_0 - \sigma p_1). \]

The above equation can then be used to eliminate one of the unknowns from the results. To determine the last unknown, another well-known technique will be used as explained further in the paper.

As we are interested in the distribution of the buffer contents, let us now define \( U(z) \) as the pgf of the buffer contents at the beginning of an arbitrary slot in the steady state. The pgf \( U(z) \) can be derived from the above results as
\[ U(z) = P(1, 1, z) = U_0(1, 1) + U_1(1, 1, z) \]
\[ = \frac{A(z)[1 - z]}{1 - A(z)} \left[ R_0(z)\tilde{\alpha}_0 + R_1(z)\tilde{\alpha}_1 \right]. \]

As easily observed from this expression, the pgf \( U(z) \) is a simple function of \( R_0(z) \) and \( R_1(z) \). Substituting then (29) and (30) and making also use of the equation (35), we get the following explicit expression for \( U(z) \), containing only the one left unknown \( p_0 + p_1 \), being the probability of an empty system:
\[ U(z) = \frac{1}{\sigma \Omega_{\cdot\cdot}(z) \Omega_0(z) + \sigma_\cdot \cdot \Omega_{\cdot\cdot}(z) \Omega_1(z)} \times \left[ (p_0 + p_1)(1 - \gamma) \cdot [1 - A(z)] \right) \]
\[ \left. \frac{1 - zA(z)^{s+1}}{\gamma \cdot [1 - A(z)]} \right]. \]
\[
\cdot \left[ (1-z)\gamma^s A(z)^{s+1} \bar{\sigma}_0 \bar{\epsilon}_1 - z(1-\gamma^s A(z)^{s+1})(\bar{\sigma}_0 + \sigma \bar{\epsilon}_1) \right] \\
+ (1-\gamma^s+1) \left[ A'(1) + \frac{1}{s+1} (p_0 + p_1 - 1)(\bar{\sigma}_0 + \sigma \bar{\epsilon}_1) \right] \\
\cdot [1 - \gamma A(z)] \cdot [1 - A(z)^{s+1}] \cdot z. 
\] 
\] 
(37)

It is shown [1], [2] that using Rouche's theorem, the remaining unknown \( p_0 + p_1 \) can be found. The procedure is as follows. Observe that \( U(z) \) is a rational function of \( z \) and denote its denominator by \( D(z) = \bar{\sigma} \Omega_1(z) \Omega_2(z) + \sigma \Omega_0(z) \Omega_1(z) \), and its numerator by \( N(z) \), such that \( U(z) = N(z)/D(z) \). As \( U(z) \) is a probability generating function, it has the property of being analytic inside the unit disc, i.e., for \( \{ z : |z| \leq 1 \} \). This means that it cannot have any poles lying in this area. Suppose there exists a value \( z^* \) of \( z \) for which the denominator of (37) is equal to zero and which lies inside the unit disc. Then \( z^* \) cannot be a pole of \( U(z) \), and therefore, the numerator of (37) also has to vanish for \( z = z^* \). So, any zero of the denominator for which \( |z| \leq 1 \) is also a zero of the numerator. Now, finding any individual zero of the denominator (where all parameters are known), which lies in the unit disc, each time leads to the construction of one extra equation. This equation is built in the way that the numerator of (37) is

\[ N(z^*) = 0, \]

for every individual zero \( z^* \). So, in summary, after identifying \( z^* \) as the zero of the denominator in a numerical way, this value can then be used in (38) to solve for the final unknown \( p_0 + p_1 \).

4.1 Moments of the buffer contents

Now that we have the distribution of the buffer contents in the form of its pgf \( U(z) \), we can derive various interesting characteristics from it, such as the expected value \( E[u] \). According to the moment generating property of probability generating functions, this value is given by \( U'(1) \). After some lengthy manipulations, \( U'(1) \) is found from (37) as

\[ U'(1) = A'(1) \left( \frac{\gamma}{2} - \frac{1}{1 - \gamma} \right) \]

\[ + \frac{1}{2(1-\gamma^s+1)} \left( \bar{\sigma} \bar{\epsilon}_0 + \sigma \bar{\epsilon}_1 - (s+1)A'(1) \right) \cdot \left\{ 2\gamma^s+1 \right\} \\
\cdot \left[ (p_0 + p_1) \bar{\sigma}_0 \bar{\epsilon}_1 - (\bar{\epsilon}_0 - (s+1)A'(1))(\bar{\epsilon}_1 - (s+1)A'(1)) \right] \\
+ A'(1) \left( \frac{p_0 + p_1}{1 - \gamma} \right) \left( \bar{\sigma}_0 + \sigma \bar{\epsilon}_1 \right) \left[ (s+2)(\gamma - \gamma^s+1) + s(\gamma^s+2-1) \right] \\
+ (1-\gamma^s+1)(s+1) \left[ A''(1) - (s+2)A'(1)^2 + 2A'(1) \right]. \]

Using the same property, it is also possible to obtain results for the higher-order moments of the buffer contents up to any order, at least in principle. For instance, the variance could be obtained as \( Var[u] = U''(1) - U'(1)^2 + U'(1) \). Note that the calculation of the \( n \)th order moment involves differentiating the pgf \( U(z) \) \( n \) times and evaluating it for \( z = 1 \). Therefore, it is clear that the subsequent derivations to \( z \) will rapidly become very complex, although they do not pose any further theoretical difficulties.

4.2 Tail behaviour of buffer contents

In the previous paragraph, we have explained how to use the pgf \( U(z) \) of the buffer contents to obtain expressions for the moments of its distribution, and more specifically, its expected value. However, from a designer’s point of view, it is important to possess information regarding the asymptotic behaviour of this distribution as well, i.e., the probability that the buffer contents exceeds a certain value \( n \), when \( n \) is large. This tail distribution too, can directly be deduced from the pgf \( U(z) \) as given in (37). To do so, we use a very accurate approximation technique explained in [2], which is based on the partial fraction expansion of \( U(z) \). Specifically, from the inversion formula for \( z \)-transforms, it follows that the mass function \( u(n) = \text{Prob}[u = n] \) can be expressed as a weighted sum of negative \( n \)th powers of the poles of \( U(z) \). Since \( U(z) \) is analytic within the unit disc, all these poles have modulus larger than 1. Then, for sufficiently large \( n \), \( u(n) \) is clearly dominated by the contribution of the pole \( z_0 \) with the smallest modulus. It is shown in [2] that this “dominant” pole \( z_0 \) must necessarily be real and positive in order to ensure a non-negative mass function \( u(n) \). As such, the probability of having more than \( n \) packets in the buffer at the beginning of an arbitrary slot can be expressed by the following geometric form:

\[ \text{Prob}[u > n] \equiv -\frac{\bar{\theta}}{\bar{\theta} - 1} \left( \frac{1}{\bar{\theta}} \right)^{n+1}, \]

where \( \bar{\theta} \) is the residue of \( U(z) \) in the point \( z = z_0 \).

To identify \( \bar{\theta} \) and \( z_0 \), we can proceed as follows. Let us again write \( U(z) \) as \( N(z)/D(z) \). The dominant pole \( z_0 \) can then be calculated numerically as the smallest value larger than 1 satisfying \( D(z_0) = 0 \), using for instance the Newton-Raphson scheme. Next, assuming that \( z_0 \) has multiplicity 1, the residue \( \bar{\theta} \) can be determined from (37) as

\[ \bar{\theta} = \text{Res}_{z_0} U(z) = \lim_{z \to z_0} U(z)(z - z_0) = \frac{N(z_0)}{D'(z_0)}. \]
Figure 4: The mean buffer contents versus the mean arrival rate for three different arrival distributions and various feedback delays (s = 2, 4, 6).

Figure 5: The mean buffer contents for three different feedback delays and for two correlation factors (Bernoulli, σ = 0.2).

5. NUMERICAL EXAMPLES

Let us consider some examples to illustrate the results we obtained. First, we investigate the impact of the feedback delay (expressed in slots) on the mean buffer contents, as well as that of the variance of the arrival distribution A(z). As in all further plots in this section, we choose the channel error probabilities to be e_0 = 0.1 and e_1 = 0.8. For Fig. 4 in particular, the probability of being in the BAD state is σ = 0.2 and the slot-to-slot correlation is quite weak as the chosen correlation factor is K = 2 (recall that for a static error channel, we would have K = 1). The plot shows the mean buffer contents as given by (39), for different feedback delays (s = 2, 4, and 6) and for three different arrival distributions: Bernoulli, Poisson and geometric ones. As one could expect, the number of packets residing in the buffer is growing as the feedback delay s is getting larger. The utilisation of the channel is then getting very low. Three groups of curves converge to three different asymptotes as for every individual group (corresponding to s = 2, 4 and 6), the throughput as given by (9) has a different value (around 0.25, 0.15 and 0.1 respectively). Within these groups, the distinction between the curves is due only to the linear contribution of A'(1) in (39), which corresponds directly to the variance of the arrival distribution. As the geometric distribution has the highest variance (whereas Bernoulli the lowest) of the three considered distributions, it is expected that it will yield the highest values for E[u]. This can indeed be observed from the plot.

To study the impact of the error correlation in the transmission channel on the mean buffer contents, we choose much higher values of K. The results from the appropriate analysis are shown in Fig. 5, where E[u] is plotted again for s = 2, 4 and 6 but now for K = 10 and 100. As for all further plots, we take σ = 0.2 and a Bernoulli arrival distribution. It is clearly seen that the mean buffer contents is growing rapidly as K jumps from value 10 to 100. The phenomenon is observed even more distinctly when the feedback delay is relatively short.

In order to further investigate this observation, a more detailed analysis for a specific feedback delay (s = 2) was performed for K = 1, 2, 5, 10, 50 and 100. The results are illustrated in Figs. 6 and 7. Fig. 6 shows E[u] plotted against the system load (i.e. A'(1)/η), whereas Fig. 7 is a logarithmic plot showing the tail distribution Prob[u > n] of the buffer contents, as given by (40). In both these figures and in Fig. 8 as well, some results obtained from simulations (according to the described model, the evolution of the transmitter buffer was simulated over a few million slots) are also included. As one may observe, they coincide with the analytical results.

We also see that the possible correlated nature of the errors in the channel can drastically change the behaviour of the buffer contents, both its mean value and its tail. The larger the correlation factor, the bigger the buffer oc-
occupancy. Note that there is almost no difference between the case for \( K = 2 \) (weak correlation) and for \( K = 1 \) (uncorrelated case). However, one should take into account the well-known fact that traffic over a wireless channel has a “bursty” character. In practice, values of the correlation factor beyond 100 are very common.

Using Little’s theorem, our analysis of the mean buffer contents \( E[u] \) also provides us with the mean packet delay \( E[d] = E[u]/A(1) \). The appropriate results are shown in the Figs. 8 and 9.

Obviously, as illustrated in the Fig. 8, the delay increases significantly as the correlation factor \( K \) is getting larger. For a given load (even for a very small one), the difference in the buffer contents for different values of \( K \) can be quite big, like several or a few dozen times the value for the non-correlated case. Therefore, one can conclude that in the analysis of ARQ protocols like Stop-and-Wait and more complex ones, it is extremely important to take into account the effect of possible error correlation, especially there when the studied protocol is working in a wireless environment. Moreover, as the conditions of the wireless channel are changing, one can expect certainly some variations in the packet delay. When the channel is quite noisy (represented in our analysis by the BAD period), it is then more probable that retransmissions of erroneous packets will take place. Furthermore, when the errors are strongly correlated, it is very likely that those retransmissions will occur during the same noisy periods. This, in effect, considerably increases the mean packet delay. The proper research was carried out on the influence of the relative time the system spends in the noisy conditions on the mean packet delay. As depicted in Fig. 9, the longer the cycle of a BAD period (noisy period), the larger the packet delay. This especially can have an important impact when the round trip delay is not short anymore.

6. CONCLUSIONS

In this paper the analysis of the Stop-and-Wait protocol operating in a wireless environment has been presented. The time varying nature of the wireless transmission channel has been modelled by means of the two-state Gilbert-Elliot model. By use of a three-dimensional probability generating functions approach, the explicit formula for the generating function of the queue length in the transmitter buffer has been achieved. The influence of the error correlation on the mean buffer contents, the tail distribution of the buffer contents and the mean packet delay has then been investigated. As obtained results have shown, neglecting the error correlation effect (static channel assumption) can cause buffer sizes to be underestimated.

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Figure 8: The influence of the correlation factor $K$ on the mean packet delay (Bernoulli, $\sigma = 0.2$, $s = 2$).

References


