Queueing analysis of the stop-and-wait ARQ protocol in a wireless environment

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Abstract

This paper presents an analytical approach for analyzing the queue length and the packet delay in the transmitter buffer of a system working under a stop-and-wait retransmission protocol. The buffer at the transmitter side is modeled as a discrete-time queue with an infinite storage capacity. The numbers of packets entering the buffer during consecutive slots are assumed to be independent and identically distributed (i.i.d.) random variables. The information packets are sent through an unreliable and non-stationary channel, which is modeled by means of a two-state Markov chain. An explicit formula is derived for the probability generating function (pgf) of the system contents. This pgf can then be used to derive several queue-length characteristics as well as the mean packet delay. By means of some numerical examples the effect of error correlation on the system performance is illustrated. Finally, the obtained analytical results are compared with the appropriate simulations according to the described model.

Keywords: Automatic Repeat Request (ARQ), Discrete time queueing model; Markovian channels

1 Introduction

Automatic repeat request (ARQ) protocols are widely used to ensure reliability for transmitted data. Nowadays, when the demand for a wireless transmission is getting larger, they are even of greater importance than ever. Obviously, the wireless channel is much noisier than a classical wired channel and because of this fact the errors occurring during data transmissions take place more often. Using then ARQ protocols one may control these errors so the data delivered to the end-user is error-free.

The working idea of this kind of protocol is as follows. The transmitter sends a packet (code-word) consisting of information bits and error detection code. The receiver checks the transmitted packet by a simple decoding technique. If no error is encountered, a positive acknowledgement (ACK) is sent to the transmitter via the feedback channel and the packet is delivered to the end user. Otherwise, if an error is detected the negative acknowledgement (NACK) is sent back to the transmitter and the erroneous packet is discarded. The transmitter, upon the receipt of the NACK message, simply retransmits the packet. The time, which has elapsed from the transmission of the packet till the moment when the corresponding feedback message arrives, is called the round trip delay (RTD) and is expressed in slots (RTD is equal to \( s+1 \) slots). One slot is the fraction of time, which is necessary to transmit exactly one packet.

Generally, there are three basic ARQ schemes: Stop-and-Wait (SW), Go-back-N (GBN) and Selective Repeat (SR) [10]. In this paper the SW protocol is analyzed. As opposed to the two others protocols where the data are transmitted continuously, in the SW protocol the transmitter has to wait for acknowledgement messages (before transmitting a next packet) from the receiver (see Figure 1). In this case one can talk about wasted capacity of the channel. Despite the fact that it has a relatively low throughput, the analysis of the SW protocol under the assumption of correlated errors gives us a good physical insight into more sophisticated ARQ systems as well. Furthermore, the complexity of a system, which is using the SW protocol, is a very slight one. The SR protocol for instance, although the most efficient one, requires a resequencing buffer at the receiver side, causing the whole system to be more complex. Taking all these things into account and the nature of the wireless environment, which is used to transmit data, the SW protocol seems to be a correct choice to analyze the influence of error correlation on ARQ performance.
In this study, we mainly focus on the analysis of the queue length in the transmitter buffer. Specifically, we derive an analytical technique to obtain explicit expression for the probability generating function (pgf) of the buffer contents. From this pgf, several performance characteristics, such as the mean queue length, the tail distribution of the queue length and the mean packet delay can be derived.

The outline of the paper is as follows. The system under study and the modelling assumptions are described in Section 2. The details of the mathematical model of the system are explained in the Section 3. Section 4 deals with the queueing analysis of the system. In Section 5 some numerical examples are presented and the effect of error correlation on the system performance is discussed. Finally, in section 6, the conclusions are given.

2 System Description

We model the transmitter buffer of a system working under a Stop-and-Wait retransmission protocol as a discrete-time queue with an infinite storage capacity. The numbers of packets entering the buffer during consecutive slots are assumed to be independent and identically distributed (i.i.d.) random variables with common pgf $E(z)$. The information in the form of packets is sent to a receiver over a wireless channel. A packet is kept in the buffer as long as the information about its successful delivery (ACK message) is not received (back) by the transmitter. The feedback message that arrives at the transmitter at slot $k$ is received for a packet that was transmitted at slot $k-s$. The parameter $s$ is called the feedback delay here. The acknowledgement messages are assumed to be always error free. This assumption is justified by the fact that ACK/NACK messages are much shorter than the information packets and are very well protected once coded.

The time-varying nature of the wireless channel is modelled by a two-state Markov chain (see Figure 2), often called Gilbert-Elliot model [6, 7]. The states are termed GOOD (state 0) and BAD (state 1) and evidently model two different conditions of the channel. The probability that the transmitted packet is received with error in a GOOD and BAD state is $e_0$ and $e_1$ respectively, where $e_0 < e_1$, although this restriction is not a requirement for the analysis. The GOOD state represents a relatively satisfactory condition of the wireless channel (where the error-free transmission of a packet is highly probable), whereas in the BAD state, the proper transmission of a packet is more difficult.

At any slot, a transition from one state to another occurs with probabilities $1-\alpha$ and $1-\beta$ respectively, whereas the probabilities of the channel remaining in the same state (as analysing its evolution from slot to slot), are given by $\alpha$ (remaining in the GOOD state) and $\beta$ (remaining in the BAD state). The parameters $\alpha$ and $\beta$ could then be used as the error correlation indicators. However, the use of more comprehensive parameters is more advisable here. Let us introduce two other parameters defined

$$\sigma = \frac{1-\alpha}{2-\alpha-\beta}, \quad K = \frac{1}{2-\alpha-\beta}. \quad (1)$$

The factor $K$ can be seen as a measure for the absolute lengths of the sojourn times (in state 0 and 1), whereas the parameter $\sigma$ characterises their relative lengths. Furthermore, the factor $K$ could be called
the correlation factor as the correlation coefficient $\gamma$ between the channel states (channel errors) in two consecutive slots is determined as a simple function of $K$:

$$\gamma = \alpha + \beta - 1 = 1 - \frac{1}{K}.$$  \hspace{1cm} (2)

3 Mathematical model and system equations

In order to study the behaviour of the transmitter buffer, let us define first the random variable $u_k$ as the system contents (i.e., the number of packets or data blocks in the transmitter buffer), at the beginning of slot $k$. In view of the system and channel model described above, it is clear that the set $\{u_k\}$ does not form a Markov chain (it is not possible to determine the probability distribution of $u_{k+1}$ based on the value of $u_k$), and some extra variables are required to describe the state of the system at the start of slot $k$. Therefore, we introduce the random variable $m_k$ as follows: $m_k$ indicates the remaining number of slots needed to receive an acknowledgement message for the packet currently under transmission at the beginning of slot $k$, if $m_k \geq 1$, and $m_k = 0$ if $u_k = 0$. In the sequel, the variable $m_k$ will be referred to as the residual service time at the beginning of slot $k$. Note that when $m_k = 1$, it means that the acknowledgement message (ACK or NACK) arrives at the transmitter at the end of slot $k$, and the packet either leaves the buffer or is retransmitted (see Figure 3). Also, we add a third variable $r_k$, which represents the channel state (0 or 1) during slot $k-s$. Then, the triple $(r_k, m_k, u_k)$ constitutes a three-dimensional Markovian state description of the system at the beginning of slot $k$.

The behaviour of the system can be described by the following system equations, where $e_k$ denotes the number of packets entering the system during slot $k$:
(a) If \( m_k = 0 \) (and hence \( u_k = 0 \)):

\[
\begin{align*}
    u_{k+1} &= e_k; \\
    m_{k+1} &= \begin{cases} 
        0 & \text{if } e_k = 0, \\
        s + 1 & \text{if } e_k > 0.
    \end{cases}
\end{align*}
\]

That is, the packets in the system at the beginning of slot \( k+1 \) are only those which entered the system in slot \( k \). In case of new arrivals to the system during slot \( k \), a packet will be transmitted during the next slot \( k+1 \) and the residual service time becomes \( s+1 \) slots; otherwise the system remains empty.

(b) If \( m_k = 1 \) (and hence \( u_k > 0 \)):

\[
\begin{align*}
    u_{k+1} &= u_k - v_k + e_k; \\
    m_{k+1} &= \begin{cases} 
        0 & \text{if } v_k = u_k = 1 \text{ and } e_k = 0, \\
        s + 1 & \text{otherwise}.
    \end{cases}
\end{align*}
\]

Here the random variable \( v_k \) represents the type of acknowledgement that is received at the end of slot \( k \), i.e. \( v_k = 1 \) in case of an ACK and \( v_k = 0 \) in case of a NACK. The distribution of \( v_k \) is completely determined by the value of \( r_k \), i.e. if at slot \( k-s \) the channel was in state \( r_k = i \) \((i = 0, 1)\), the probability of receiving a NACK (ACK) for the transmitted packet at slot \( k \) is \( \text{Prob}[v_k = 0|r_k = i] = e_i \) and \( \text{Prob}[v_k = 1|r_k = i] = 1 - e_i \) respectively. When a positive acknowledgement is received \((v_k = 1)\), the packet leaves the system at the end of slot \( k \); otherwise, when due to transmissions errors a NACK message comes back, the packet must be retransmitted and stays in the system. The system becomes empty when in case of an ACK the packet that leaves at the end of slot \( k \) is the last one in the system and no new packets arrive to the system in slot \( k \); otherwise the transmission of a new packet (in case of an ACK is received) or a retransmission (in case of a NACK) starts. Note that the slots with \( m_k = 1 \) are the only slots where the channel state \( r_k \) is of influence to the evolution of the system. In other words, the decision whether a packet is transmitted correctly or not depends on the channel state in only one slot. This can indeed be seen as a limitation of the model, since one could argue that a packet is likely to travel through the wireless medium during many subsequent slots before reaching the receiver.

(c) If \( m_k > 1 \) \((m_k = 2, \ldots, s+1)\) (and hence \( u_k > 0 \)):

\[
\begin{align*}
    u_{k+1} &= u_k + e_k; \\
    m_{k+1} &= m_k - 1,
\end{align*}
\]

i.e. the packet does not leave the system at the end of slot \( k \), so the evolution of the system occupancy is only due to the new arrivals occurring during slot \( k \); the residual service time is simply decreased by one in this case.

4 Steady-state queueing analysis

4.1 Generating function of the buffer contents

In this section, we will derive an expression for the pgf of the system contents at the beginning of an arbitrary slot in the steady state.

Naturally the steady state exists only when the average number of packets entering the system is strictly less than the throughput of the stop-and-wait ARQ protocol. Note that an expression for the throughput of the analysed system is different from the classical one (when a static channel is considered). As the time varying nature of the transmission channel is taken into account, the expression for the throughput takes the following form:

\[
\eta = (1-\sigma) \cdot \frac{1-e_0}{1+s} + \sigma \cdot \frac{1-e_1}{1+s}.
\]
Our first step is to define the joint pgf $P_k(x, y, z)$ of the state vector $(r_k, m_k, u_k)$ as

$$P_k(z, y, z) = E[x^{r_k} y^{m_k} z^{u_k}] = \sum_{i=0}^{1} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \text{Prob}[r_k=i, m_k=j, u_k=n] x^i y^j z^n, \quad (10)$$

where the operator $E[\ldots]$ stands for the expected value of the expression between the brackets.

Next, we derive an expression for the function $P_{k+1}(x, y, z)$, the joint pgf at the beginning of slot $k+1$. Based on the system equations (3)–(8), the pgf $P_{k+1}(x, y, z)$ is calculated as follows:

$$P_{k+1}(x, y, z) = E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}}]$$

$$= \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k=0] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}} | r_k=i, m_k=0]$$

$$+ \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k=1] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}} | r_k=i, m_k=1]$$

$$+ \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k>1] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}} | r_k=i, m_k>1]. \quad (11)$$

As our channel is modelled by means of a two-state Markov chain, it is clear that its evolution process can be described by the two following pgfs:

$$T_0(x) = E[x^{r_{k+1}} | r_k=0] = \alpha + (1-\alpha)x; \quad T_1(x) = E[x^{r_{k+1}} | r_k=1] = 1 - \beta + \beta x. \quad (12)$$

Now, by making use of the equations (4), (6), (8) and (12), the equation (11) can be further rewritten as:

$$P_{k+1}(x, y, z) = \sum_{i=0}^{1} \left\{ \text{Prob}[r_k=i, m_k=0] \text{Prob}[e_k=0] T_i(x) \right. \right.$$

$$+ \text{Prob}[r_k=i, m_k=0] \text{Prob}[e_k>0] E[z^{e_k} | e_k>0] y^{s+1} T_i(x)$$

$$+ \text{Prob}[r_k=i, m_k=1, u_k=1] \text{Prob}[e_k=0] T_i(x) (1-e_i)$$

$$+ \text{Prob}[r_k=i, m_k=1, u_k=1+e_k>0] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}+e_k} | r_k=i, m_k=1, u_k=1+e_k>0] (1-e_i)$$

$$+ \text{Prob}[r_k=i, m_k=1] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}+e_k} | r_k=i, m_k=1] e_i$$

$$+ \text{Prob}[r_k=i, m_k>1] E[x^{r_{k+1}} y^{m_{k+1}} z^{u_{k+1}+e_k} | r_k=i, m_k>1]\right\}. \quad (13)$$

Moving ahead with the calculations, we get:

$$P_{k+1}(x, y, z) = \left[ E(0) + (E(z) - E(0)) y^{s+1} \right] \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k=0, u_k=0] T_i(x)$$

$$+ E(0) \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k=1, u_k=1] T_i(x) (1-e_i)$$

$$+ y^{s+1} \sum_{i=0}^{1} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \text{Prob}[r_k=i, m_k=1, u_k=n] \text{Prob}[e_k=j] z^{n+1+j} T_i(x) (1-e_i)$$

$$- y^{s+1} E(0) \sum_{i=0}^{1} \text{Prob}[r_k=i, m_k=1, u_k=1] T_i(x) (1-e_i)$$

$$+ y^{s+1} E(z) \sum_{i=0}^{1} \sum_{n=1}^{\infty} \text{Prob}[r_k=i, m_k=1, u_k=n] z^n T_i(x) e_i$$

$$+ \frac{1}{y} E(z) \sum_{i=0}^{1} \sum_{j=2}^{\infty} \sum_{n=1}^{\infty} \text{Prob}[r_k=i, m_k=j, u_k=n] y^j z^n T_i(x). \quad (14)$$
The equation (14) can be simplified further and finally, we obtain

\[ P_{k+1}(x, y, z) = \sum_{i=0}^{1} T_i(x) \left\{ \frac{P[x, 0, 0]}{y} \left[ E(0) + (E(z) - E(0))y^{i+1} - \frac{E(z) - E(0)}{y} \right] \right. \]

\[ + R_i(0)(1 - e_i) E(0)(1 - y^{i+1}) \]

\[ + E(z) R_i(z) [y^{i+1}(1 - e_i) + z y^{i+1}e_i - z] \]

\[ + \frac{1}{y} E(z) U_i(y, z) \left\} \right., \quad (15) \]

where the partial generating functions \( R_i(z) \) and \( U_i(y, z) \) are defined as:

\[ R_i(z) = \sum_{n=1}^{\infty} \text{Prob}[r_i = i, m_k = 1, u_k = n] z^{n-1}; \quad (16) \]

\[ U_i(y, z) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \text{Prob}[r_i = i, m_k = j, u_k = n] y^j z^{n-1}. \quad (17) \]

In the steady state, the functions \( P_k(x, y, z) \), \( R_i(z) \) and \( U_i(y, z) \) become independent of \( k \) and converge to some limiting functions \( P(x, y, z) \), \( R_i(z) \) and \( U_i(y, z) \):

\[ P(x, y, z) = \lim_{k \to \infty} P_k(x, y, z); \quad R_i(z) = \lim_{k \to \infty} R_i(z); \quad U_i(y, z) = \lim_{k \to \infty} U_i(y, z). \quad (18) \]

By suppressing the indices \( k \) in equation (15), we then get the following expression for the pgf \( P(x, y, z) \) in terms of the partial probability generating functions \( U_0(y, z) \) and \( U_1(y, z) \):

\[ P(x, y, z) = \sum_{i=0}^{1} T_i(x) \left\{ \frac{p_i}{y} \left[ E(0) + (E(z) - E(0))y^{i+1} - \frac{E(z) - E(0)}{y} \right] \right. \]

\[ + R_i(0)(1 - e_i) E(0)(1 - y^{i+1}) \]

\[ + E(z) R_i(z) [y^{i+1}(1 - e_i) + z y^{i+1}e_i - z] \]

\[ + \frac{1}{y} E(z) U_i(y, z) \left\} \right., \quad (19) \]

where the following notation was used: \( p_i = \text{Prob}[i, 0, 0] \). It is clear that the sum of the two probabilities \( p_0 \) and \( p_1 \) (i.e., \( p_0 + p_1 \)) is the probability of having an empty system.

From the definitions (10) and (17), it follows that \( P(x, y, z) \) can also be expressed as:

\[ P(x, y, z) = U_0(y, z) + U_1(y, z) x. \quad (20) \]

Identification of the coefficients of equal powers of \( x \) on the right-hand sides of equations (19) and (20) then yields the following set of two linear equations for \( U_0(y, z) \) and \( U_1(y, z) \):

\[ U_0(y, z)[y - \alpha E(z)] = (1 - \beta) E(z) U_1(y, z) \]

\[ + [E(0)y + (E(z) - E(0))y^{i+2} - E(z)] \left[ p_0(1 - \beta) \right] \]

\[ + E(0)y(1 - y^{i+1})(1 - e_0) R_0(0) + (1 - \beta)(1 - e_1) R_1(0) \]

\[ + y E(z) \left[ \alpha R_0(z)(y^{i+1} - z) + (1 - \beta) R_1(z)(y^{i+1} - z) \right]; \quad (21) \]

\[ U_1(y, z)[y - \beta E(z)] = (1 - \alpha) E(z) U_1(y, z) \]

\[ + [E(0)y + (E(z) - E(0))y^{i+2} - E(z)] \left[ p_0(1 - \alpha) + p_1 \right] \]

\[ + E(0)y(1 - y^{i+1})(1 - e_0) R_0(0) + \beta(1 - e_1) R_1(0) \]

\[ + y E(z) \left[ (1 - \alpha) R_0(z)(y^{i+1} - z) + \beta R_1(z)(y^{i+1} - z) \right] , \]
where the functions $\phi_i(z)$ are defined as

$$\phi_i(z) = e_i z + (1-e_i).$$

(22)

Now, solving for the two functions $U_0(y, z)$ and $U_1(y, z)$ gives

$$U_0(y, z)[y - E(z)][y - \gamma E(z)] = \left[ E(0)y + (E(z) - E(0))y^{s+2} - E(z) \right] \left\{ y[p_0\alpha + p_1(1-\beta)] - \gamma E(z)p_0 \right\}$$

$$+ E(0)y(1-y^{s+1}) \left\{ A[y - \beta E(z)] + BE(z)(1-\beta) \right\}$$

$$+ yE(z) \left\{ [y\alpha - \gamma E(z)]R_0(z)(y^{s+1}\phi_0(z) - z) + y(1-\beta)R_1(z)(y^{s+1}\phi_1(z) - z) \right\};$$

(23)

$$U_1(y, z)[y - E(z)][y - \gamma E(z)] = \left[ E(0)y + (E(z) - E(0))y^{s+2} - E(z) \right] \left\{ y[p_0(1-\alpha) + p_1\beta] - \gamma E(z)p_1 \right\}$$

$$+ E(0)y(1-y^{s+1}) \left\{ B[y - \alpha E(z)] + AE(z)(1-\alpha) \right\}$$

$$+ yE(z) \left\{ (1-\alpha)R_0(z)(y^{s+1}\phi_0(z) - z) + [y\beta - \gamma E(z)]R_1(z)(y^{s+1}\phi_1(z) - z) \right\},$$

where the notations $A$, $B$ and $\gamma$ are defined as:

$$A = (1-\alpha)(1-e_0)R_0(0) + (1-\beta)(1-e_1)R_1(0);$$

(24)

$$B = (1-\alpha)(1-e_0)R_0(0) + \beta(1-e_1)R_1(0);$$

(25)

$$\gamma = \alpha + \beta - 1.$$

(26)

The next step is of course to determine the two unknown functions $R_0(z)$ and $R_1(z)$ and the four unknown parameters $p_0$, $R_0(0)$, $p_1$ and $R_1(0)$ in order to obtain the complete expression for the pgf $P(x, y, z)$.

From the system equations (3)–(8), we easily observe that $m_k = 0$ if and only if $u_k = 0$. This observation gives rise to the following two equalities:

$$U_0(y, 0) = U_0(0, 0) = p_0, \quad \text{for all } y;$$

(27)

$$U_1(y, 0) = U_1(0, 0) = p_1, \quad \text{for all } y.$$

(28)

If we now set $z=0$ in the equations (23) and invoke the above equalities, we find the following relationships between $p_0$, $p_1$, $R_0(0)$ and $R_1(0)$:

$$p_0 + p_1 = \frac{E(0)}{1-E(0)}[(1-e_0)R_0(0) + (1-e_1)R_1(0)];$$

(29)

$$(1-\beta)p_1 - (1-\alpha)p_0 = \frac{\gamma E(0)}{1-\gamma E(0)}[(1-e_1)(1-\beta)R_1(0) - (1-e_0)(1-\alpha)R_0(0)].$$

(30)

The second step towards the determination of the unknowns from the equations (23) is another very useful remark concerning the partial probability generating functions $U_0(y, z)$ and $U_1(y, z)$. It is clear namely that these functions should be bounded for all values of $y$ and $z$ such that $|y| \leq 1$ and $|z| \leq 1$. Naturally this is also true for $y_1 = E(z)$ or $y_2 = \gamma E(z)$ and $|z| \leq 1$, since $|E(z)| \leq 1$ and $|\gamma E(z)| \leq 1$ for all $z$ with $|z| \leq 1$, since $E(z)$ is a pgf.

If we choose $y = y_1$ or $y = y_2$ in the set of equations (23), where $|z| \leq 1$, the left-hand sides of these equations vanish. Evidently, the right-hand sides of (23) also have to be equal to zero. This leads to a set of equations for the functions $R_0(z)$ and $R_1(z)$, from which these functions are finally obtained as

$$R_0(z) = \frac{(1-\beta)\Omega_{1, 2}(z)|1-E(z)|E(z)^{s}[p_0 + p_1] - \Omega_1(z)E(z)^{s}[1-\gamma E(z)]([1-\beta)p_1 - (1-\alpha)p_0]}{(1-\beta)\Omega_{1, 2}(z)\Omega_0(z) + (1-\alpha)\Omega_0, \Omega_1(z)};$$

(31)

$$R_1(z) = \frac{(1-\alpha)\Omega_{0, 2}(z)|1-E(z)|E(z)^{s}[p_0 + p_1] - \Omega_0(z)E(z)^{s}[1-\gamma E(z)]([1-\alpha)p_0 - (1-\beta)p_1]}{(1-\beta)\Omega_{1, 2}(z)\Omega_0(z) + (1-\alpha)\Omega_0, \Omega_1(z)},$$

(32)
where we have introduced the following notations:

\[ \Omega_1(z) = E(z)^{s+1} \phi_1(z) - z; \]
\[ \Omega_{\gamma}(z) = E(z)^{s+1} \gamma^{s+1} \phi_1(z) - z. \]  

(33)  
(34)

Note that to derive the equations (31) and (32) the definitions (making (39) shorter):

Now, as one can observe the only remaining unknowns in the equations for \( R_0(z) \) and \( R_1(z) \) (and also in the equations for \( U_0(y, z) \) and \( U_1(y, z) \)) are the probabilities \( p_0 \) and \( p_1 \). One relationship between the two unknowns is simply found from the normalisation condition for \( P(x, y, z) \):

\[ P(1, 1, 1) = U_0(1, 1) + U_1(1, 1) = 1. \]  

(35)

This results in

\[ E'(1) = R_0(1)(1-e_0) + R_1(1)(1-e_1), \]

(36)

where \( E'(1) \) is the mean arrival rate. After some lengthy manipulations, involving multiple applications of de l'Hôpital's rule, the following, more explicit form of (36) can be found:

\[ (1-\gamma^{s+1}) \left( (s+1)E'(1) + (p_0+p_1-1)[1-(1-\sigma)e_0-\sigma e_1] \right) = \gamma^s(1-\gamma)(s+1)(e_0-e_1)(\sigma p_0 - (1-\sigma)p_1). \]  

(37)

The above equation can then be used to eliminate one of the unknowns from the results. To determine the last unknown another well-known technique will be used as explained further in the paper.

As we are interested in the distribution of the buffer occupancy, let us now define \( U(z) \) as the probability generating function of the buffer contents at the beginning of an arbitrary slot in the steady state. The pgf \( U(z) \) can be derived from the above results as

\[ U(z) = P(1, 1, z) = U_0(1, z) + U_1(1, z) \]
\[ = \frac{E(z)}{1-E(z)[1-\gamma E(z)]}\left\{ [1-\gamma E(z)][\phi_0(z) - z]R_0(z) + [1-\gamma E(z)][\phi_1(z) - z]R_1(z) \right\} \]
\[ = \frac{E(z)}{1-E(z)} \left\{ [\phi_0(z) - z]R_0(z) + [\phi_1(z) - z]R_1(z) \right\} = \frac{E(z)(1-z)}{1-E(z)} \left[ R_0(z)(1-e_0) + R_1(z)(1-e_1) \right]. \]

(38)

The equation (38) as easily observed is the function of \( R_0(z) \) and \( R_1(z) \). Substituting then (31) and (32) and making also use of the equation (37) results in the following explicit expression for \( U(z) \), containing only the one left unknown \( p_0 + p_1 \), being the probability of the empty system:

\[ U(z) = \frac{1}{\sigma^2 \Omega_1(\gamma)(z) \Omega_0(z) + \sigma^2 \Omega_0(z) \Omega_1(z)} \cdot \frac{(1-z)E(z)^{s+1}}{(1-\gamma)[1-E(z)]} \]
\[ \cdot \left\{ (1-\gamma^{s+1})[1-\gamma E(z)][1-E(z)^{s+1}]z [E'(1) + \frac{1}{s+1}(p_0+p_1-1)(\bar{\sigma} e_0 + \sigma e_1)] \right\} \]
\[ + (p_0 + p_1)(1-\gamma)[1-E(z)][(1-\gamma)\gamma^{s+1}E(z)^{s+1}\bar{e}_0 \bar{e}_1 - z (1-\gamma^{s+1}E(z)^{s+1})(\bar{\sigma} e_0 + \sigma e_1)] \right\}, \]

(39)

where we have used the following definitions (making (39) shorter):

\[ \bar{\sigma} \equiv 1-\sigma \quad \text{and} \quad \bar{e}_i \equiv 1-e_i. \]  

(40)

Note that in equation (39) the set of two parameters \( \alpha \) and \( \beta \) was replaced by another set of two, more comprehensive parameters \( \sigma \) and \( \gamma \) (or \( \sigma \) and \( K \) as the correlation coefficient \( \gamma \) is determined just only by the value of \( K \)) as discussed at the beginning of this paper.

It can be shown [1, 2] that using Rouche's theorem, the remaining unknown could be found. The procedure is as follows. Observe that \( U(z) \) is a rational function of \( z \) and let us denote its denominator by \( D(z) = \sigma \Omega_1(\gamma)(z) \Omega_0(z) + \sigma \Omega_0(z) \Omega_1(z) \), and its numerator by \( N(z) \), such that \( U(z) = N(z)/D(z) \). As \( U(z) \) is a probability generating function, it has the property of being analytic inside the unit disc, i.e. for \( \{ z : |z| < 1 \} \). This means that it cannot have any poles lying in this area. Suppose there exists a value \( z^* \) of \( z \) for which the denominator of (39) is equal to zero and which lies inside the unit disc. Then
\( z^\# \) cannot be a pole of \( U(z) \), and therefore, the numerator of (39) also has to vanish for \( z = z^\# \). Based on the fact that \( U(z) \) has no poles inside the unit disc, any zero of the denominator for which \( |z| \leq 1 \) is also a zero of the numerator. Now, finding any individual zero of the denominator (where all parameters are known), which lies in the unit disc, each time leads to the construction of one extra equation. This equation is built in the way that the numerator of (39) is

\[
N(z^\#) = 0, \quad (41)
\]

for every individual zero \( z^\# \).

So, in summary, after identifying \( z^\# \) as the zero from the denominator in a numerical way, this value can then be used in (41) to solve for the final unknown \( p_0 + p_1 \).

### 4.2 Moments of the buffer contents

Now that we have the distribution of the buffer contents in the form of its pgf \( U(z) \), we can derive various interesting characteristics from it, such as the expected value \( E[u] \). According to the moment generating property of probability generating functions, this value is given by \( U'(1) \), where the prime symbol denotes the first derivative of \( U(z) \) with respect to \( z \). After some lengthy manipulations, \( U'(1) \) is found from (39) as

\[
U'(1) = E'(1) \left( \frac{s}{2} - \frac{\gamma}{1 - \gamma} \right) + \frac{1}{2(1 - \gamma^{s+1})[\sigma \bar{e}_0 + \sigma \bar{e}_1 - (s+1)E'(1)]} \left\{ 2\gamma^{s+1} \left[ (p_0 + p_1)\bar{e}_0\bar{e}_1 - (\bar{e}_0 - (s+1)E'(1))(\bar{e}_1 - (s+1)E'(1)) \right] + E'(1) \frac{p_0 + p_1}{1 - \gamma} (\sigma \bar{e}_0 + \sigma \bar{e}_1) \left[ (s+2)(\gamma - \gamma^{s+1}) + s(\gamma^{s+2} - 1) \right] + (1 - \gamma^{s+1})(s+1)[E''(1) - (s+2)E'(1)^2 + 2E'(1)] \} \right. \quad (42)
\]

Using the same property, it is also possible to obtain results for the higher-order moments of the buffer contents up to any order, at least in principle. For instance, the variance could be obtained as \( \text{Var}[u] = U''(1) - U'(1)^2 + U'(1) \). Note that the calculation of the \( n \)th order moment involves differentiating the pgf \( U(z) \) \( n \) times and evaluating it for \( z = 1 \). Therefore, it is clear that the subsequent derivations to \( z \) will rapidly become very complex, although they do not pose any further theoretical difficulties.

### 4.3 Tail behaviour of the buffer contents

In the previous paragraph, we have explained how to use the pgf \( U(z) \) of the buffer contents to obtain expressions for the moments of its distribution, and more specifically, its expected value. However, from a designers point of view, it is important to possess information regarding the asymptotic behaviour of this distribution as well, i.e. the probability that the buffer contents exceeds a certain value \( n \), when \( n \) is large. This tail distribution too, can directly be deduced from the pgf \( U(z) \) as given in (39). To do so, we use a very accurate approximation technique explained in [2], which is based on the partial fraction expansion of \( U(z) \).

Specifically, from the inversion formula for \( z \)-transforms, it follows that the mass function \( u(n) = \Pr[u = n] \) can be expressed as a weighted sum of negative \( n \)th powers of the poles of \( U(z) \). Since \( U(z) \) is analytic within the unit disc, all these poles have modulus larger than 1. Then, for sufficiently large \( n \), \( u(n) \) is clearly dominated by the contribution of the pole \( z_0 \) with the smallest modulus. It is shown in [2] that this dominant pole \( z_0 \) must necessarily be real and positive in order to ensure a non-negative mass function \( u(n) \). As such, the probability of having more than \( n \) packets in the buffer at the beginning of an arbitrary slot can be expressed by the following geometric form:

\[
\Pr[u > n] \cong - \frac{\theta}{z_0 - 1} \left( \frac{1}{z_0} \right)^{n+1}, \quad (43)
\]

where \( \theta \) is the residue of \( U(z) \) in the point \( z = z_0 \).
To identify \( \theta \) and \( z_0 \), we can proceed as follows. Let us again write \( U(z) \) as \( N(z)/D(z) \). The dominant pole \( z_0 \) can then be calculated numerically as the smallest value larger than 1 satisfying \( D(z_0)=0 \), using for instance the Newton-Raphson scheme. Next, assuming that \( z_0 \) has multiplicity 1, the residue \( \theta \) can be determined from (39) as

\[
\theta = \text{Res}_{z=z_0} U(z) = \lim_{z \to z_0} U(z)(z-z_0) = \frac{N(z_0)}{D'(z_0)},
\]

where \( N(z_0) \) can be obtained directly from (39) by evaluating the numerator for \( z = z_0 \). Also, after expanding the functions (33) and (34) in \( D(z) \), we find

\[
D'(z_0) = 2(1-\gamma)\gamma^{s+1}E(z_0)^{2(s+1)} \left[ (s+1) \frac{E'(z_0)}{E(z_0)} \phi_0(z_0) \phi_1(z_0) + \frac{e_0+e_1}{2} - (1-z_0)e_0e_1 \right] + 2(1-\gamma)z_0 - (1-\gamma)(1+\gamma^{s+1})E(z_0)^{s+1} \left[ 1 + z_0(s+1) \frac{E'(z_0)}{E(z_0)} \right] - (1-\gamma)E(z_0)^{s+1} \left[ z_0 - (1-z_0) \left( 1 + z_0(s+1) \frac{E'(z_0)}{E(z_0)} \right) \right] \left[ \sigma(e_0 + \gamma^{s+1}e_1) + \sigma(e_1 + \gamma^{s+1}e_0) \right].
\]

## 5 Numerical examples

Let us consider some examples to illustrate the results we obtained. First, we investigate the impact of the round trip delay (expressed in slots) on the mean buffer contents, as well as that of the variance of the arrival distribution \( E(z) \). As in all further plots in this section, we choose the channel error probabilities to be \( e_0=0.1 \) and \( e_1=0.8 \). For Figure 4 in particular, the probability of being in the BAD state is \( \sigma=0.2 \) and the slot-to-slot correlation is quite weak as the chosen correlation factor is \( K=2 \) (recall that for a static error channel, we would have \( K=1 \)). The plot shows the mean buffer contents as given by (42), for different round trip delays (\( s=2, 4, \) and 6) and for three different arrival distributions: Bernoulli, Poisson and geometric ones. As one could expect, the number of packets residing in the buffer is growing as the round trip delay \( s+1 \) is getting larger. The utilisation of the channel is then getting very low. Three groups of curves converge to three different asymptotes as for every individual group (corresponding to \( s=2, 4 \) and 6), the throughput as given by (9) has a different value (around 0.25, 0.15 and 0.1 respectively). Within these groups, the distinction between the curves is due only to the linear
contribution of $E'(1)$ in (42), which corresponds directly to the variance of the arrival distribution. As the geometric distribution has the highest variance (whereas Bernoulli the lowest) of the three considered distributions, it is expected that it will yield the highest values for $E[u]$. This can indeed be observed from the plot.

To study the impact of the error correlation in the transmission channel on the mean buffer contents, we choose much higher values of $K$. The results from the appropriate analysis are shown in Figure 5, where $E[u]$ is plotted again for $s = 2, 4$ and $6$ but now for $K = 10$ and $100$. As for all further plots, we take $\sigma = 0.2$ and a Bernoulli arrival distribution.

It is clearly seen that the mean buffer contents is growing rapidly as the $K$ jumps from value 10 to 100. The phenomenon is observed even more distinctly when the round trip delay is relatively short. In order to further investigate this observation, a more detailed analysis for a specific RTD ($s = 2$) was performed for $K = 1, 2, 5, 10, 50$ and 100. The results are illustrated in figures 6 and 7. Figure 6 shows $E[u]$ plotted against the system load (i.e. $E'(1)/\eta$), whereas Figure 7 is a logarithmic plot showing the tail distribution $\text{Prob}[u > n]$ of the buffer contents as given by (43). In both these figures and in Figure 8 as well, some results obtained from simulations (according to the described model, the evolution of the transmitter buffer was simulated over few millions slots) are also included. As one may observe, they coincide with the results obtained from (39).

Now, as one observes, the possible correlated nature of the errors in the channel can drastically change the behaviour of the buffer, both its mean value and its tail. The larger the correlation factor the bigger the buffer occupancy. Note that there is almost no difference between the case for $K = 2$ (weak correlation) and for $K = 1$ (uncorrelated case). However, one should take into account the well-known fact that traffic over a wireless channel has a bursty character. In practice, values of the correlation factor beyond 100 are very common.

Using the Little’s theorem, our analysis of the mean buffer contents $E[u]$ also provides us with the mean packet delay $E[d] = E[u]/E'(1)$. The appropriate results are shown in the figures 8 and 9.

Obviously, as illustrated in the Figure 8, the delay increases significantly as the correlation factor $K$ is getting larger. For a given load (even for a very small one), the difference in the system occupancies for different values of $K$ can be quite big, like several or a few dozen times the value for a non-correlated case. Therefore, one can conclude that in the analysis of ARQ protocols like Stop and Wait and more complex ones, it is extremely important to take into account possible error correlation effect, especially there when studied protocol is working in the wireless environment. Moreover, as the conditions of the wireless channel are changing, one can expect certainly some variations in the packet delay. When
the channel is quite noisy (represented in our analysis by the BAD period), it is then more probable that retransmissions of erroneous packets will take place. Furthermore, when the errors are strongly correlated, it is very likely that those retransmissions will occur during the same noisy periods. This, in effect, considerably increases the mean packet delay. The proper research was carried out on the influence of the relative time the system spends in the noisy conditions on the mean packet delay. As depicted in the Figure 9, the longer the cycle of a BAD period (noisy period) the larger the packet delay. This especially can have an important impact when the round trip delay is not short anymore.

6 Conclusions

In the paper the analysis of the stop and wait protocol operating in a wireless environment has been presented. The time varying nature of the wireless transmission channel has been modelled by means of the two-state Gilbert-Elliot model. By the use of a three-dimensional probability generating functions approach the explicit formula for the generating function of system contents has been achieved. The influence of the error correlation on the mean queue length, tail distribution of the buffer contents and the mean packet delay has been then investigated. As obtained results have shown, the negligence of the error correlation effect (static channel assumption) can cause buffer sizes to be underestimated.

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References


