Analysis of the interdeparture process in consecutive stages of a VoIP network

Veronique Inghelbrecht, Bart Steyaert, Sabine Wittevrongel and Herwig Bruneel

SMACS Research Group, Department of Telecommunications and Information Processing, Ghent University, Sint-Pietersnieuwstraat 41, B-9000 Gent, Belgium

Abstract

In this paper, we will investigate the evolution of the interarrival and interdeparture times between voice packets when they are proceeding a number of network nodes. But, instead of separately specifying the characteristics of each individual source, we will model the arrival process in a node as the superposition of a single tagged stream and an independent background process that aggregates the remaining traffic sources. Because we assume that the load of a single voice stream is very low compared to the load of the aggregate traffic, we can represent the tagged voice packets as markers (packets with size zero).

At the entrance of each network node we thus have the tagged traffic stream, namely the markers, and the background stream. The tagged marker stream is characterized by the consecutive interarrival times between the markers, which are assumed to be identically distributed. The background arrival process, on the other hand, is described on a slot-per-slot basis according to a general i.i.d. process. First, we will establish an expression for the probability generating function (pgf) of the interdeparture time of the voice packets after one stage. We will use this interdeparture-time pgf as the pgf of the interarrival time of the voice packets in the next stage, in order to assess the evolution of the interarrival-time characteristics throughout the network. Also, we will calculate the joint pgf of the interdeparture times between 3 voice packets (in case two successive interarrival times may be dependent of each other). To end this paper, we will present some numerical results and draw some conclusions.

1 Introduction

In voice over IP-networks, it may be possible that the voice packets have to pass a number of network nodes before reaching their final destination. In this paper we will
investigate the end-to-end delay of a tagged voice stream. To obtain the end-to-end delay characteristics a convolution technique is often used, which means that all queueing processes in the subsequent nodes of the network are assumed to be independent. In [1], experimental results are presented to check whether consecutive delays are statistically independent. Another technique, which is used in [2-5], applies a proper parameter fitting method to the input and output process.

In this paper, we will investigate a voice stream that has to pass a number of network nodes. At the entrance of each node, we have the tagged voice stream and a background process that aggregates the remaining traffic. (In [6-8], it is revealed in which circumstances this kind of aggregation of a number of sources into one background process will lead to accurate predictions of the system performance.) Because the load of the tagged voice stream is very low compared to the total load arriving at each node, we can represent the tagged voice stream as markers. In order to obtain the distribution of the interarrival time between two markers at the entrance of a certain node, we will calculate the distribution of the interdeparture time at the output of the previous node.

A summary of this paper is as follows. In the next section, we will give a detailed description of the model. In section 3, we derive system equations for the interdeparture time between two packets. Expressions for the mean and variance of the interdeparture time are given in section 4. In section 5, we will calculate the joint generating function of the interdeparture times between 3 successive voice packets. Finally, some numerical examples are given in section 6, and conclusions are formulated in section 7.

2 The model

We assume a discrete-time model, meaning that time is slotted and one slot suffices for the transmission of exactly one packet (i.e., one unit of information). We consider a tagged stream that has to pass a number of nodes. The buffer of each of the nodes has an infinite storage capacity, i.e., no arriving packets are lost. For the rest of this section, we focus attention on one particular node.

At the entrance of each node we have a tagged stream, which generates markers (packets of size zero). The Interarrival Time (IAT) between two successive markers, i.e., the number of slots between their slots of arrival, is a random variable, in the sequel denoted by $I_n$. The $I_n$'s are assumed to be identically distributed according to the probability mass function $i(k)$. The corresponding probability generating function (pgf) is denoted by $I(z)$.

Furthermore, there is some additional background traffic entering each node. We consider the case that this Background Arrival Process (BAP) is described by the pgf $A(z)$ during any slot, i.e., the sequence of random variables describing the numbers of packet arrivals generated by the background stream during consecutive slots are assumed to be independent and identically distributed (i.i.d), and independent of the tagged marker stream. We assume that the background packets are of length one, and hence, have one
slot of service time. We define the load of the BAP traffic and thus the load of all the traffic entering the node (the tagged stream has load zero) as
\[ \rho = A'(1) \ . \]
All packets are served according to a FCFS (First-Come-First-Served) discipline and there is a single server serving the packets entering the node. We also suppose that the markers arrive at the slot boundaries. This means that, if a marker arrives at the beginning of slot \( m \) all the BAP packets arriving before slot \( m \) are served before the marker. If the marker arrives in a non-empty buffer, it will leave at the same time instant as the last BAP packet that arrived before slot \( m \), otherwise the marker leaves the buffer at the end of slot \( m \).

3 The interdeparture time at the output of a node

In this section, we will calculate the pgf of the interdeparture time between two successive markers at the output of a node (using the pgf of the interarrival time between the two markers and the pgf describing the background arrival process at the entrance of the node).

Let us consider two arbitrary successive markers (say the \( n \)-th and the \( n + 1 \)-th). We denote by \( I_n \) respectively \( \tilde{I}_n \) (with pgf \( I_n(z) \) respectively \( \tilde{I}_n(z) \)) the number of slots between the arrival instants respectively the departure instants of the two tagged markers. Furthermore, we define \( d_n \) (with pgf \( D_n(z) \)) as the delay of the \( n \)-th tagged marker entering the buffer.

The relationship between \( I_n \), the interarrival time at the entrance of a node, and \( \tilde{I}_n \), the interdeparture time at the output of a node, (which is also the interarrival time between two nodes at the entrance of the next stage) is given by (see Figure 1)
\[ \tilde{I}_n = d_{n+1} - d_n + I_n \ . \]

![Figure 1: Relationship between the random variables \( d_n \), \( I_n \), \( d_{n+1} \) and \( \tilde{I}_n \)](image)

We suppose that the \( n \)-th marker arrives at the beginning of slot zero. If we call \( v_m \) the system contents at the beginning of slot \( m \), i.e., the number of BAP packets in the buffer at the beginning of slot \( m \), we can express the delay of the \( n \)-th marker as :
\[ d_n = (v_0 - 1)^+ + 1 \ , \]

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where \((...)^+\) denotes \(\max\{0,\ldots\}\). If the \(n\)-th marker arrives in slot zero, the \(n+1\)-th marker arrives in slot \(I_n\). Thus we obtain for the delay of the \(n+1\)-th marker:

\[
d_{n+1} = (v_n - 1)^+ + 1
\]  

(3)

Let \(a_m\) (with pgf \(A(z)\)) denote the number of background arrivals during slot \(m\). The relationship between \(v_{m+1}\) (with pgf \(V_{m+1}(z)\)) and \(v_m\) (with pgf \(V_m(z)\)) is given by:

\[
v_{m+1} = (v_m - 1)^+ + a_m
\]

(4)
or,

\[
(v_m - 1)^+ = v_{m+1} - a_m
\]

Due to the uncorrelated nature of the background arrival process, the above system equation can be expressed in terms of pgf see [9]) as,

\[
V_{m+1}(z) = \frac{A(z)[V_m(z) + (z-1)V_m(0)]}{z}
\]

(5)

Expressing \(V_{m+1}(z)\) in terms of \(E[z^{(\alpha-1)^+}]\), we obtain

\[
V_{m+1}(z) = \frac{A(z)^{m+1}}{z^m} E[z^{(\alpha-1)^+}] + (z-1) \sum_{j=0}^{m} \left( \frac{A(z)}{z} \right)^{j+1} V_{m-j}(0)
\]

or,

\[
E[z^{v_{m+1}}] = E\left[ \sum_{k=0}^{m} a_{k-m} (z-1)^{a_{k-m}} \right] + (z-1) \sum_{j=0}^{m} E\left[ \sum_{k=m-j}^{m} a_{k-j-1} \{v_{m-j} = 0\} \right]
\]

(6)

where the notation

\[
E[\ldots;\{v_j = 0\}] \triangleq \text{Prob}[v_j = 0]E[\ldots;v_j = 0]
\]

We will now write the pgf \(\tilde{I}_n(z)\) in function of the known pgf's \(A(z)\) and \(I_n(z)\). From (1)-(4), we have

\[
\tilde{I}_n(z) = E[z^{\tilde{I}_n}] = E[z^{d_{n+1-d_n}+I_n}]
\]

\[
= E[z^{v_{m+1}+a_{m}-(\alpha-1)^++I_n}]
\]

(7)

Combining equations (6) and (7), we obtain

\[
\tilde{I}_n(z) = I_n(A(z)) + (z-1) \sum_{k=0}^{\infty} \sum_{j=0}^{k-1} A(z)^j z^{k-j-1} E[z^{-(\alpha-1)^+}\{v_{k-j} = 0\}]
\]

(8)
If we define
\[ P_j(z) \triangleq E \left[ z^{(m-1)^+} \{ v_j = 0 \} \right] , \]
we can rewrite equation (8) as:
\[ \tilde{I}_n(z) = I_n(A(z)) + (z - 1) \sum_{j=1}^{\infty} z^{-1} P_j(z^{-1}) \sum_{k=j}^{\infty} i(k) A(z)^{k-j} . \]

Equation (9) gives an expression for the pgf of the interdeparture time at the output of the node in terms of the pgf of the interarrival time at the input of the node and the pgf of the number of BAP arrivals in a slot. The only unknown functions in this expression are the \( P_j(z) \). We will now determine these functions.

First of all, we remark that \( P_0(z) = \text{Prob}(v_0 = 0) \). Using equation (6), we get
\[
E \left[ z^{(m-1)^+} x^{v_j} \right] = E \left[ z^{(m-1)^+} x^{(m-1)^+ + \sum_{i=0}^{j-1} a_i + 1} \right] \\
+ (x - 1) \sum_{l=1}^{j-1} E \left[ z^{(m-1)^+} x^{k-l + \sum_{i=l}^{j-1} a_i} \{ v_l = 0 \} \right].
\]

If we set \( x = 0 \) in this equation, we obtain the following recursive formula for \( P_j(z) \):
\[
P_j(z) = E \left[ z^{(m-1)^+} \{ (v_0 - 1)^+ = j - 1 - \sum_{i=0}^{j-1} a_i \} \right] \\
+ \sum_{l=1}^{j-1} \left( \text{Prob} \left( \sum_{i=l}^{j-1} a_i = j - 1 - l \right) - \text{Prob} \left( \sum_{i=l}^{j-1} a_i = k - j \right) \right) P_l(z),
\]

where
\[
E \left[ z^{(m-1)^+} \{ (v_0 - 1)^+ = k - 1 - \sum_{j=0}^{k-1} a_j \} \right] = \\
\sum_{j=0}^{k-1} z^j \text{Prob}((v_0 - 1)^+ = j) \text{Prob} \left( \sum_{j=0}^{k-1} a_j = k - 1 - j \right).
\]

Finally, we will determine the distribution \( q(i) = \text{Prob}((v_0 - 1)^+ = i) \) in the steady state. We can reach this steady state if the number of BAP packets that arrives during a slot is less then the number of BAP packets that can be served per slot, i.e., if
\[ A'(1) = \rho < 1 \]
We can write the following equation for the pgf of the system contents $V(z)$ in the steady state (see equation (5))

$$V(z) = \frac{(z-1)V(0)}{z - A(z)}.$$ 

Using this equation and the normalization condition we obtain

$$v(0) = 1 - A'(1) = 1 - \rho \quad ;$$

$$q(0) = \frac{1 - \rho}{a(0)}.$$ 

Using equation (4), we can derive the following recursive formula for the distribution $q(k)$:

$$q(0) = q(0)(a(0) + a(1)) + q(1)a(0)$$

$$\Rightarrow q(1) = \frac{(1 - a(0) - a(1))q(0)}{a(0)}$$

$$q(k - 1) = \sum_{j=0}^{k-1} q(j)a(k - j) + q(k)a(0) \quad k \geq 2$$

$$\Rightarrow q(k) = \frac{(1 - a(1))(1 - a(0)) + \sum_{j=0}^{k-2} q(j)a(k - j)}{a(0)} \quad k \geq 2 . \quad (12)$$

Equation (12) gives a recursive formula for the distribution $q(k)$.

Using equations (9)-(12), we can derive the whole distribution of the interdeparture time at the output of the node and thus of the IAT at the entrance of the next stage of the network. So, if we have the distributions of the interarrival times between two successive markers in one stage, we can calculate the distributions of the interarrival times in the other stages of the network.

4 Mean and variance of the interdeparture time

Taking the first derivative of (6), we obtain the mean interarrival time in the next stage of the network:

$$\bar{I}'_n(1) = I'_n(1)A'(1) + \sum_{k=0}^{\infty} i_n(k) \sum_{j=0}^{k-1} \text{Prob}(v = 0)$$

$$= I'_n(1) .$$

So, as could be expected the mean interarrival time stays constant if the tagged marker is proceeding in the network. We define $I$ the mean interarrival time:

$$I = I'_n(1) = I'_n(1) . \quad (13)$$
Taking the second derivative of $\tilde{I}_n(z)$ and using $\text{var}[\tilde{I}_n] = I_n^* (1) + I^2$, we obtain the variance of the interarrival time at the input of the next stage of the network.

\[
\text{var}[\tilde{I}_n] = I A''(1) + I_n''(1) A'(1)^2 + I - I^2 + \\
2 \sum_{k=0}^{\infty} i_n(k) \sum_{j=0}^{k-1} \left( j A'(1) \text{Prob}(v = 0) + (k - j - 1) \text{Prob}(v = 0) - P'_{k-j}(1) \right)
\]

\[
= I A''(1) + I_n''(1) A'(1)^2 + I - I^2 + \\
\sum_{k=0}^{\infty} i_n(k) \left( k(k - 1) A'(1) \text{Prob}(v = 0) + k(k - 1) \text{Prob}(v = 0) - 2 \sum_{j=0}^{k-1} P'_{k-j}(1) \right)
\]

\[
= I A''(1) + \text{var}[I_n] - 2 \sum_{k=0}^{\infty} i_n(k) \sum_{j=1}^{k} P'_j(1) .
\] (14)

5 Correlation between two successive interdeparture times

In this section, we will investigate the evolution of the correlation between two successive interarrival times. Therefore, at the entrance of a node, we consider three successive markers (marker $n$, $n + 1$ and $n + 2$). Given the joint distribution of their interarrival times $I_n$ and $I_{n+1}$ (We suppose that these interarrival times may be dependent of each other), we will calculate the joint pgf of their interdeparture times.

Using Figure 2, we obtain

\[
E \left[ x^{I_n} y^{I_{n+1}} \right] = E \left[ x^{I_n-(n-1)+} \left( \frac{x}{y} \right)^{(v_n-1)+} y^{I_{n+1}+(v_{n+1}-1)+} \right] .
\]

We will use an analogous method as in the previous sections to obtain this joint gener-
ating function. We then subsequently get
\[
E \left[ x^{I_n} y^{I_{n+1}} \right] = E \left[ x^{I_n} A(y)^{I_{n+1}} \right] + (y - 1) \sum_{j=1}^{\infty} y^{j-1} \sum_{k=0}^{j} x^k.
\]
\[
E \left[ x^{-(v_0-1)^+} \left( \frac{x}{y} \right)^{(v_{k+1}-1)^+} \{v_{k+1} = 0\} \right] \sum_{l=j}^{\infty} \text{Prob}(i_n = k, i_{n+1} = l) A(y)^{l-j}
= E \left[ A(x)^{I_n} A(y)^{I_{n+1}} \right] + (y - 1) \sum_{j=1}^{\infty} x^{j-1} P_j(x^{-1}) \sum_{k=j}^{\infty} A(x)^{k-j}.
\]
\[
\sum_{i=0}^{\infty} \text{Prob}(i_n = k, i_{n+1} = l) A(y)^l + (y - 1) \sum_{j=1}^{\infty} y^{j-1} \sum_{k=0}^{j} x^k.
\]
\[
E \left[ x^{-(v_0-1)^+} \left( \frac{x}{y} \right)^{(v_{k+1}-1)^+} \{v_{k+1} = 0\} \right] \sum_{l=j}^{\infty} \text{Prob}(i_n = k, i_{n+1} = l) A(y)^{l-j} \tag{15}
\]

In this equations, the only unknown functions are the $P_{mn}(x, y)$, defined as:
\[
P_{mn}(x, y) \triangleq E \left[ x^{(v_0-1)^+} y^{(v_m-1)^+} \{v_{m+n} = 0\} \right]. \tag{16}
\]

We can rewrite this function as:
\[
P_{mn}(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m+i} \text{Prob}((v_m - 1)^+ = i, v_{m+n} = 0) \cdot
\]
\[
\frac{\text{Prob}((v_0 - 1)^+ = j, (v_m - 1)^+ = i)}{q(i)} x^i y^j \tag{17}
\]

The first factor in this equation is known because we know $P_n(z)$. The second factor in this equation follows from
\[
E \left[ x^{(v_0-1)^+} y^{v_m} \right] = E \left[ x^{(v_0-1)^+} y^{(v_m-1)^+ + \sum_{i=0}^{m-1} \alpha_i} \right]
\]
\[
+ (y - 1) \sum_{l=1}^{m-1} E \left[ x^{(v_0-1)^+} y^{m-l + \sum_{i=0}^{m-1} \alpha_i} \{v_l = 0\} \right]
\]
\[
y^{-m+1} Q(xy) A(y)^m + (y - 1) \sum_{l=1}^{m-1} \left( \frac{A(y)}{y} \right)^{m-l} P_l(x) \tag{18}
\]

Equations (15)-(18) give the joint pgf of two successive interdeparture times at the output of a node. This means that if we have the joint pgf of the interarrival times at the entrance of a node, we can calculate the joint pgf and thus the joint distribution $\text{Prob}(i_n = k, i_{n+1} = l)$ of the interarrival times at the entrance of the next stage. We can also derive an expression for the covariance of the interarrival times at the next stage:
\[
cov[\bar{I}_n, \bar{I}_{n+1}] = E \left[ (\bar{I}_n - I)(\bar{I}_{n+1} - I) \right]
\]
\[
= \frac{\text{var}[\bar{I}_n + \bar{I}_{n+1}] - \text{var}[\bar{I}_n] - \text{var}[\bar{I}_{n+1}]}{2}.
\]

8
6 Numerical results

In this section, we will concentrate on the case where at the entrance of the first stage of the network the tagged marker stream is periodic. This means that at the first stage of the network, the pgf of the interarrival time between two markers is given by

\[ I(z) = z^I \quad \text{with } I \geq 1 \quad . \tag{19} \]

Throughout this section, we have chosen a load \( \rho = 0.8 \) for the background arrival process.

![Graph](image_url)

Figure 3: Variance of the interarrival time in function of the number of passed stages, for \( A(z) = 0.1z^2 + 0.6z + 0.3 \).

In Figures 3 and 4, we have plotted the variance of the interarrival times in different stages of the network, for different values of \( I \), the interarrival time at the first stage of the network (which also equals the mean interarrival time). For Figure 3, we have chosen a BAP process with the following pgf:

\[ A(z) = 0.1z^2 + 0.6z + 0.3 \quad , \tag{20} \]

which means that in each slot there are either two, one or zero BAP arrivals. In Figure 4, we have chosen a Poisson process for the BAP process with pgf

\[ A(z) = e^{(0.8(z-1))} \quad . \tag{21} \]
Figure 4: Variance of the interarrival time in function of the number of passed stages, for 
$A(z) = e^{0.8(z-1)}$.

This figure reveals that although the mean interarrival time $I$ remains constant when the markers traverse different nodes, the variance of the interarrival times seems to increase almost linearly with the number of nodes the markers have passed. Note that the first two terms of equation (14) constitute an upper bound for the variance of the interarrival times, which is also plotted in the Figures 3 and 4 and is quite close for low values of the mean interarrival time $I$, i.e.,

$$\text{var}[\bar{I}_n] \approx I A''(1) + \text{var}[I_n] \quad .$$

Using this formula, we can see that the variance of the interarrival times depends on the second-order derivative of the pgf of the BAP arrival process and, more specifically, we can expect the interarrival-time variance to be higher in the case of a Poisson BAP process then in the case of $A(z) = 0.1z^2 + 0.6z + 0.3$. This is confirmed by the curves in Figures 3 and 4.

Figures 5 and 6 show the probability that the interdeparture time between two markers exceeds a given value $X$ at the output of stages 1, 2, 4, 8, 16, 32, in the case that the interarrival time $I$ at the entrance of the first node equals 4 slots. Again we have chosen as BAP the process described by the pgf of (20) (in Figure 5) and a Poisson BAP process (in Figure 6). From these figures, it is clear that as the number of stages traversed by the markers increases, the probability of having an interdeparture time equal to zero gets higher, and (in view of the constant mean interdeparture time) the tail of the interdeparture-time distribution becomes longer. This observation is also in accordance
with the fact that if the interarrival time at the entrance of a node is zero, the interdeparture time at the output of the node will be zero too. We also observe that the tails are longer in case of a Poisson BAP process than in the case of a BAP process with pgf given by equation (20).

In Figure 7, we have plotted the covariance of two successive interdeparture times after one stage under the assumption that at the entrance of the stage the tagged stream is periodic with period $I$ (and hence, the interarrival times are independent). We have again considered the BAP processes described by the pgf's of equations (20) and (21). We see that the covariance is negative, which means that two successive interdeparture times are negatively correlated. This also means that the variance of the sum of two successive interdeparture times is less than the variance of each of the individual interdeparture times.

Finally, in Figure 8, the covariance of two successive interarrival times is shown as a function of the number of stages traversed by the markers. As BAP, the process described by the pgf of equation (20) is considered, and the mean interarrival time $I$ equals 1,2,4 and 8 slots. Again, we observe that we obtain an approximately linear relationship between the number of stages passed by the markers and the covariance.
Figure 6: The tail distribution of the interdeparture time, for $A(z) = e^{0.8(z-1)}$ and $I = 4$.

7 Conclusions

In this paper, we have studied the characteristics of the interdeparture time between two markers of a tagged stream at the output of a network node in case that the marker stream is disturbed by a background process that aggregates the remaining traffic and is independent of the tagged marker stream. We have derived expressions for the pgf of the interdeparture time as well as for the joint pgf of two successive interdeparture times. By using the distribution of the interdeparture time at the output of a node as the distribution of the interarrival time in the next node, we have investigated the evolution of the interarrival times between markers as they proceed in the network. Our results indicate that the interarrival time between markers remains constant throughout the network, and that both the variance of the interarrival time and the covariance between two successive interarrival times are approximately linear functions of the number of stages traversed by the markers. Also, we have observed that if at the entrance of the first network stage two successive interarrival times are independent of each other, after one stage they become negatively correlated.

References

Figure 7: The covariance of two successive interdeparture times after one stage in function of $I$, the interarrival time between two markers at the entrance of the first node.


Figure 8: The covariance of two successive interarrival times in function of the number of passed stages, for $A(z) = 0.1z^2 + 0.6z + 0.3$ and $I = 1, 2, 4, 8$
