Characteristics of switched traffic in a multistage ATM network

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Abstract

In this paper, a method is developed to derive the traffic characteristics of the switches belonging to consecutive stages of an ATM switching network. We will develop a generic model for characterizing the traffic sources in consecutive network stages, based on the assumption that the queuing processes in subsequent network nodes are independent. In our model, the parameters that determine the arrival process on the inlets of a switch in a stage solely depend on the output process on the outlets of the switches belonging to the previous stage. As a result, we will be able to calculate the mean value and variance, as well as the whole distribution of the system contents and the cell delay in a tagged switch, anywhere in the network, which is subject to future study. Furthermore, we will prove that, under the assumptions of the model, these characteristics converge to a limiting value after a few stages in the network, independent of the characteristics of the traffic sources at the network access point.

1 Introduction

Multistage networks are strong candidates for implementation of ATM switching fabrics in Broadband ISDN networks. To prevent internal loss of data, buffers are often used inside the switching elements of the fabric. But, while the performance of a single switch node has been exhaustively examined (e.g., [1]), the statistical behaviour of the traffic modified as it crosses the network has not been thoroughly analyzed yet, apart from some simulation studies reported in [2], [3] and the references therein. One of the bottlenecks to examine this behaviour, is the characterization of the output process of such a buffer.

The usual analytical approach to assume independence between consecutive nodes in a network, and apply a proper parameter fitting method to the input and output processes. Several studies that use this technique under various modelling assumptions were reported in [4], [5], [6], [7], [8], [9]. In this paper, we will proceed along the same lines. We will present a tractable approximation model to characterize the departure process on an outlet of a switching element by an \( n \)-state Markov Modulated Bernoulli Process (MMBP) Arrival Process, with \( n = 2 \) or 3.

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2 The Model

We consider an ATM switching network consisting of a matrix of elementary switches that provide the routing of cells from their inlets to their outlets. In this section, we focus attention on one particular switch, as depicted in Figure 1.

As in all discrete-time queueing problems, the time axis is divided into slots of fixed length, and one slot suffices for the transmission of exactly one cell. Packets enter the switch via one of its input links and are routed to one of the output links, where they are buffered in a separate output queue, corresponding to the cells’ destination, to await transmission of the cells that arrived before them for this particular output queue and that are not yet sent. The case of single-server output queues is treated here, i.e., each output queue corresponds to exactly one output link providing the transmission of the cells. In this paper, the number of input and output links of an individual switch is set to \( N \).

Since only one cell can be carried by each of the input or output links during one slot, the number of cell arrivals during a slot on each of the input links is either 0 or 1. In this paper, we assume that the arrival process describing the number of cell arrivals during each slot via one of the input links, is a time-correlated process, namely a 3-state MMBP Arrival Process, which will be described in more detail in the following paragraph.

![Figure 2: State transition diagram of the arrival process](image)

We assume that the length of a passive period (a number of consecutive slots during which there is no arrival at all) is geometrically distributed with parameter \( \beta \)

\[
\text{Prob[a passive period contains } n \text{ slots]} = (1 - \beta)\beta^{n-1}, \quad n \geq 1
\]

On the other hand, the length of an active period (a number of consecutive slots during which there is a cell arrival on the input link) is considered to be a probabilistic mixture of two geometric distributions. This means that there are two types of active periods on the input
links: an active period 1, geometrically distributed with parameter $\alpha_1$, and an active period 2, geometrically distributed with parameter $\alpha_2$

\[
\text{Prob[an active period 1 contains } n \text{ slots]} = (1 - \alpha_1)\alpha_1^{n-1}, \quad n \geq 1, \\
\text{Prob[an active period 1 contains } n \text{ slots]} = (1 - \alpha_2)\alpha_2^{n-1}, \quad n \geq 1.
\]  

(1)

During each slot of both types of active periods on an input link, there is a cell arrival. The values of $\alpha_1$, $\alpha_2$, and $\beta$ can be chosen independently. When an active period of either of both types ends, it is immediately followed by a passive period. When a passive period comes to an end, then with probability $p$ an active period 1 will start, and with probability $1 - p$ an active period 2 will start (in the following, we will also use the expressions "a cell arrival of type 1" and "a cell arrival of type 2" for a cell arriving during an active period 1 and an active period 2 respectively, although, of course, no physical distinction can be made between the two cells themselves). Under these circumstances, the pair of random variables describing the numbers of cell arrivals of type 1 and type 2 respectively on the $N$ input links during consecutive slots form a two-dimensional first-order Markov chain.

We define an arrival path as the virtual path between an input link and an output queue. Thus, $N$ arrival paths connect an output queue with the $N$ input links of the switch. If a cell arrives on an input link during a slot, it can be routed to a tagged output queue (via the arrival path that connects the input link with the tagged output queue) with probability $1/N$, since the routing of cells from the input links to the output queues of the switch is performed in a uniform and independent way, and there are $N$ possible destinations. This assumption of uniform and independent routing of cells is supported by the architecture presented in [10], for which the model was initially developed. Thus, the random variable $f$ denoting the number of cells that are routed from an input link to a tagged output link, given that a cell is generated on the input link, has probability generating function

\[
f(z) = \frac{N - 1}{N} + \frac{z}{N}.
\]

(2)

The purpose of this paper is to develop a method for deriving the values of the parameters characterizing the correlated arrival process in the switches of consecutive stages, from the parameters of the correlated arrival process in the switches of the previous stage.

3 Characterization of the correlated arrival process in one stage

As before, we consider a single switch. We will now derive the distribution of the pair of random variables describing the numbers of cell arrivals of type 1 and 2 respectively in a tagged output queue, in terms of the parameters of the arrival process on the input links. In the next section, it will become clear that this is an important quantity, if we want derive the parameters of the arrival process in subsequent stages.

The arrival process on the $N$ arrival paths connecting the $N$ input links to a tagged output queue, is completely determined by the arrival process on the $N$ input links. One can prove that, although the routing of cells from the input links to the output queues is performed in an independent and uniform way, the random variables describing the lengths of consecutive passive periods on the arrival paths are not geometrically distributed, if the random variables describing the lengths of a passive period and the two types of active periods on the input links are geometrically distributed (nevertheless, the random variables describing the lengths of an active period 1 and an active period 2 on the arrival paths are geometrically distributed under these conditions). Consequently, the pair of random variables that indicate the numbers of cell arrivals of both types on the $N$ arrival paths to a tagged output queue during consecutive slots, do not form a first-order Markov chain, thus making the derivation of an analytical solution of this system hard to obtain. Therefore, to be able to calculate the distribution for the pair of
random variables describing the numbers of cell arrivals of both types in a tagged output queue, we will assume that this pair of random variables constitute a two-dimensional homogeneous first-order Markov chain, when the system has reached its steady-state. This is equivalent to assuming that the lengths of both the passive and the active (of the two types) periods on the arrival paths are geometrically distributed. This is the case for the distribution of the two types of active periods, but an approximation for the distribution of the passive periods.

We denote by \(a_{k,i}^i\) \((i=1\ or\ 2)\) the number of cell arrivals of type \(i\) on the input link during slot \(k\) and by \(e_{k,i}\) the number of arrivals of type \(i\) at the tagged output queue. As already mentioned, each of the \(a_{k,i}\) cells generated by the input links during slot \(k\), will be sent to the tagged output queue with probability \(1/N\). Therefore, the relation between \(a_{k,i}\) and \(e_{k,i}\) is given by

\[
e_{k,i} = \sum_{j=1}^{a_{k,i}} f_j
\]

where the probability generating function corresponding to each of the random variables \(f_j\) is equal to \(f(z)\), defined in (2).

First of all, let us define \(\psi_{m,n}(x_1,x_2)\) as

\[
\psi_{m,n}(x_1,x_2) \triangleq E \left[ x_1^{a_{1,k}} x_2^{a_{2,k}} | a_{1,k-1} = m, a_{2,k-1} = n \right]
\]

which, in view of the 3-state MMBP arrival process satisfies

\[
\psi_{m,n}(x_1,x_2) = c_1(x_1)^m c_2(x_2)^n d(x_1,x_2)^{N-m-n},
\]

where

\[
c_i(x) = 1 - \alpha_i + \alpha_i x \quad i = 1, 2
\]

\[
d(x_1,x_2) = \beta + (1 - \beta) p x_1 + (1 - \beta)(1 - p)x_2
\]

This implies that we may write, due to (3)

\[
E \left[ x_1^{e_{1,k}} x_2^{e_{2,k}} | a_{1,k-1} = m, a_{2,k-1} = n \right] = E \left[ f(x_1)^{a_{1,k}} f(x_2)^{a_{2,k}} | a_{1,k-1} = m, a_{2,k-1} = n \right] = \psi_{m,n}(f(x_1), f(x_2))
\]

We thus obtain

\[
E_{k,j}(x_1,x_2) \triangleq E \left[ x_1^{e_{1,k}} x_2^{e_{2,k}} | e_{1,k-1} = i, e_{2,k-1} = j \right]
\]

\[
= \sum_{m=i}^{N} \sum_{n=j}^{N-m} \psi_{m,n}(f(x_1), f(x_2)) \text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n | e_{k-1,1} = i, e_{k-1,2} = j] (4)
\]

where we have used the property that \((e_{k,1}, e_{k,2})\) does not depend on the value of \((e_{k-1,1}, e_{k-1,2})\) if \((a_{k-1,1}, a_{k-1,2})\) is known.

The quantities \(\text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n | e_{k-1,1} = i, e_{k-1,2} = j]\) can be calculated from Bayes’ rule as follows

\[
\text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n | e_{k-1,1} = i, e_{k-1,2} = j] = \frac{\text{Prob}[e_{k-1,1} = i, e_{k-1,2} = j | a_{k-1,1} = m, a_{k-1,2} = n] \text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n]}{\sum_{m} \sum_{n} \text{Prob}[e_{k-1,1} = i, e_{k-1,2} = j | a_{k-1,1} = m, a_{k-1,2} = n] \text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n]}
\]

\[
= c_{i,j} \text{Prob}[e_{k-1,1} = i, e_{k-1,2} = j | a_{k-1,1} = m, a_{k-1,2} = n] \text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n], (5)
\]

where the constant \(c_{i,j}\) will be determined afterwards using the normalization condition.
We obtain combining (4) and (5)

\[ E_{i,j}(x_1, x_2) = c_{i,j} \sum_{m=0}^{N} \sum_{n=0}^{N-m} \text{Prob}[e_{k-1,1} = i, e_{k-1,2} = j | a_{k-1,1} = m, a_{k-1,2} = n] \]

\[ \text{Prob}[a_{k-1,1} = m, a_{k-1,2} = n] \sigma_1(f(x_1))^m \sigma_2(f(x_2))^n d(f(x_1), f(x_2))^{N-m-n} \]  \( \text{where} \) \[ \sigma_1 = \frac{p(1 - \beta)(1 - \alpha_2)}{(1 - \beta)(1 - p\alpha_2 - (1 - p)\alpha_1) + (1 - \alpha_1)(1 - \alpha_2)} \]

\[ \sigma_2 = \frac{(1 - p)(1 - \beta)(1 - \alpha_1)}{(1 - \beta)(1 - p\alpha_2 - (1 - p)\alpha_1) + (1 - \alpha_1)(1 - \alpha_2)} \]  \( \text{denotes the fraction of time an input link remains in active period 1 respectively 2, and} \)

\[ \sigma = \sigma_1 + \sigma_2 \]  \( \text{is the fraction of time an input link is active. This is also the average load of an input link and} \)

\( \text{due to the uniform and independent routing of arriving cells also the average number of cell} \)

\( \text{arrivals per slot in a tagged output queue.} \)

Finally, using the explicit expressions of the remaining probabilities in the equation (6), we find

\[ E_{i,j}(x_1, x_2) = c_{i,j} \sum_{m=0}^{N} \sum_{n=0}^{N-m} \binom{m}{i} \left( \frac{1}{N} \right)^i \left( \frac{N - 1}{N} \right)^{m-i} \binom{n}{j} \left( \frac{1}{N} \right)^j \left( \frac{N - 1}{N} \right)^{n-j} \]

\[ \binom{N}{n} \binom{N - n}{m} \sigma_1^m \sigma_2^m (1 - \sigma)^{N-m-n} \sigma_1(f(x_1))^m \sigma_2(f(x_2))^n d(f(x_1), f(x_2))^{N-m-n} \]

\( \text{where} \binom{n}{k} = \frac{n!}{k!(n-k)!}. \text{Bearing in mind that} E_{i,j}(1, 1) = 1, \text{and defining} \)

\[ \gamma_i = \frac{\sigma_i}{N - \sigma} \left( \frac{1 - \alpha_i}{N} \right) \quad i = 1, 2 \]  \( \text{this can be transformed into} \)

\[ E_{i,j}(x_1, x_2) = \]

\[ (1 - \gamma_1 - \gamma_2 + \gamma_1 x_1 + \gamma_2 x_2)^{N-i-j} \left( 1 - \frac{\alpha_1}{N} + \frac{\alpha_1}{N} x_1 \right)^i \left( 1 - \frac{\alpha_2}{N} + \frac{\alpha_2}{N} x_2 \right)^j \]  \( \text{Let us denote by} e_{i,j}(m, n) \text{the steady-state probability that there are} i \text{arrivals of type 1} \)

\( \text{and} j \text{arrivals of type 2 in the output buffer during a slot, if there were} m \text{arrivals of type 1 and} \)
$n$ arrivals of type 2 during the previous slot. From (10), it is easily verified that the transition probabilities $e_{i,j}(m,n)$ satisfy

$$e_{i,j}(m,n) = \sum_{k=(m-i)\uparrow}^{m} \sum_{l=(n-j+k)\uparrow}^{N-i-j} \binom{N-i-j}{l} \binom{1}{k} (1-\gamma_1-\gamma_2)^{N-i-j-l} \gamma_1^{i-m+k} \gamma_2^{j-n-k+l} \left(1 - \frac{\alpha_1}{N}\right)^{i-m+k} \left(1 - \frac{\alpha_2}{N}\right)^{j-n-k+l}$$

where $(\cdot,\cdot)^+ \triangleq \max(\cdot,\cdot)$ and $(\cdot,\cdot)^- \triangleq \min(\cdot,\cdot)$.

Under the above assumptions, the pair of random variables $(e_{k,1}, e_{k,2})$, describing the numbers of arrivals of type 1 and 2 on the $N$ arrival paths together during slot $k$, form a homogeneous two-dimensional first-order Markov chain, and consequently, their steady-state distribution is fully determined by the transition probabilities in (11).

4 Characterization of the correlated arrival process in consecutive stages

Let us now consider an entire switching network, consisting of multiple stages. In this section, a method is developed for deriving the values of the parameters characterizing the generalized correlated arrival process in a given switch, from the arrival process in the previous switch, under the assumption that the load

$$\sigma = \sigma_1 + \sigma_2$$

remains constant in each state. Therefore, we need a notation that indicates which stage of the network we consider. If $x$ is a parameter defined in the previous section, we will denote by $x^r$ the value of this parameter in stage $r$ of the network.

First of all, assume that the $k$-th slot on a tagged outlet of an output queue in the $r$-th stage of the network is a passive slot (this outlet, of course also is an inlet of a switch belonging to the $(r+1)$-th stage). The probability that the $(k+1)$-th slot on this outlet will be a passive slot, knowing that the $k$-th slot was a passive slot, i.e. $\beta^{r+1}$, is equal to

$$\beta^{r+1} = \text{Prob} \left[ e_{1,k}^r = e_{2,k}^r = 0 | s_k^r = 0 \right]$$

where $s_k^r$ denotes the system contents at the beginning of slot $k$ in the tagged output queue belonging to the $r$-th stage of the network and $e_{i,k}^r$, $1 \leq i \leq 2$, denote the number of cell arrivals of type $i$ in the output queue during slot $k$. Since $s_k^r$ is zero, there have been no cell arrivals during slot $k-1$. We thus find

$$\beta^{r+1} = \text{Prob} \left[ e_{1,k}^r = e_{2,k}^r = 0, e_{1,k-1}^r = e_{2,k-1}^r = 0 | s_k^r = 0 \right] .$$

This becomes

$$\beta^{r+1} = \text{Prob} \left[ e_{1,k}^r = e_{2,k}^r = 0 | e_{1,k-1} = e_{2,k-1} = 0, s_k^r = 0 \right] .$$

As explained in the previous section, the number of cell arrivals of type 1 and 2 during consecutive slots on the arrival paths to a tagged output queue is assumed to form a homogeneous two-dimensional first-order Markov chain, which implies that

$$\beta^{r+1} = \text{Prob} \left[ e_{1,k}^r = e_{2,k}^r = 0 | e_{1,k-1} = e_{2,k-1} = 0 \right] .$$

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With the notations used in the previous section, this is equal to

$$\beta^{r+1} = e_0^{r+1}(0, 0) \quad ,$$

where the quantity $e_0^{r+1}(0, 0)$ can be calculated in terms of $(\alpha_1^r, \alpha_2^r, \gamma_1^r, \gamma_2^r)$. Using (11), we obtain

$$\beta^{r+1} = (1 - \gamma_1^r - \gamma_2^r)^N \quad .$$

Now, defining $1/(1 - \alpha^r)$ as the mean length of an active period on the inlets of a switch in the $r$-th stage, this quantity must satisfy

$$\frac{1}{1 - \alpha^r} = \frac{\nu^r}{1 - \alpha_1^r} + \frac{1 - \nu^r}{1 - \alpha_2^r} \quad .$$

This yields

$$\alpha^r = \frac{\nu^r \alpha_1^r + (1 - \nu^r) \alpha_2^r - \alpha_1^r \alpha_2^r}{1 - \nu^r \alpha_2^r - (1 - \nu^r) \alpha_1^r} \quad . \tag{13}$$

Combining (7), (8), (9) and (13), we thus obtain

$$\beta^{r+1} = \left( \frac{N - 2\sigma + \frac{\nu^r \sigma}{N}}{N - \sigma} \right)^N \quad .$$

Furthermore, since the load on an inlet of a switch is a constant in each stage, this quantity must satisfy

$$\sigma = \frac{1 - \beta^r}{2 - \alpha^r - \beta^r} \quad . \tag{14}$$

Using this relation to eliminate $\alpha^r$ from the previous expression for $\beta^{r+1}$, we finally obtain

$$\beta^{r+1} = \left( \frac{N - 1}{N} \left( N + 1 - 2\sigma \right) + \frac{1 - \sigma}{N} \beta^r \right)^N \quad . \tag{15}$$

What remains now is an algorithm for calculating the values of the three parameters $(\alpha_1^{r+1}, \alpha_2^{r+1}, \beta^{r+1})$ in consecutive stages, starting from given values of these parameters in the first stage. We assume that $\beta^{r+1}$ is calculated first using (15), and thus is a known quantity. It is clear that we will need three equations relating $(\alpha_1^{r+1}, \alpha_2^{r+1}, \beta^{r+1})$ with $\beta^{r+1}$ and the parameters characterizing the correlated arrival process in stage $r$. A first relation is obtained by expressing that the load on the inlets of a switch belonging to the $(r + 1)$-th stage is equal to $\sigma$. This leads to

$$\frac{\nu^r \alpha_1^{r+1}}{1 - \alpha_1^{r+1}} + \frac{1 - \nu^r}{1 - \alpha_2^{r+1}} = \frac{\sigma}{(1 - \sigma)(1 - \beta^{r+1})} \triangleq c_0^r \quad , \tag{16}$$

where $c_0^r$ can be calculated in terms of $\sigma$ and $\beta^r$ using (15). The two remaining relations are obtained by calculating the probability that an active period on an inlet of a switch belonging to the $(r + 1)$-th stage is equal to one slot and equal to two slots respectively, in terms of the parameters $(\alpha_1^r, \alpha_2^r, \beta^r, \nu^r)$, characterizing the arrival process on the inlets of a switch belonging to the $r$-th stage

$$\nu^r \left( 1 - \alpha_1^{r+1} \right) + \left( 1 - \nu^r \right) \left( 1 - \alpha_2^{r+1} \right) \triangleq c_1^r \quad , \tag{17}$$

and

$$\nu^r \left( 1 - \alpha_1^{r+1} \right) \alpha_1^{r+1} + \left( 1 - \nu^r \right) \left( 1 - \alpha_2^{r+1} \right) \alpha_2^{r+1} \triangleq c_2^r \quad . \tag{18}$$
The calculation of \( c_1^r \) and \( c_2^r \) in terms of \( (\alpha_1^r, \alpha_2^r, \beta^r, p^r) \) will be done later.

The relations (16), (17) and (18) form a set of three non-linear equations in the unknown quantities \( (\alpha_1^{r+1}, \alpha_2^{r+1}, p^{r+1}) \).

With the first equation, we can write \( p^{r+1} \) in terms of \( \alpha_1^{r+1} \) and \( \alpha_2^{r+1} \), while using the two first equations, we write \( \alpha_2^{r+1} \) in terms of \( \alpha_1^{r+1} \)

\[
p^{r+1} = \frac{(1 - \alpha_1^{r+1}) (c_0^r - \alpha_2^{r+1} - 1)}{\alpha_1^{r+1} - \alpha_2^{r+1}} \quad (19)
\]

\[
1 - \alpha_2^{r+1} = \frac{c_1^r + \alpha_1^{r+1} - 1}{1 - c_0^r + \alpha_2^{r+1}} \quad (20)
\]

\[
c_2^r = p^{r+1} \left( 1 - \alpha_1^{r+1} \right) \alpha_1^{r+1} + \left( 1 - p^{r+1} \right) \left( 1 - \alpha_2^{r+1} \right) \alpha_2^{r+1} \quad .
\]

Finally, eliminating \( \alpha_2^{r+1} \) from this expression using (20), we obtain a quadratic equation for \( \alpha_1^r \)

\[
\left( \alpha_1^{r+1} \right)^2 (c_0^r c_1^r - 1) + \alpha_1^{r+1} (2 - c_1^r - c_0^r c_1^r - c_0^r c_2^r) + c_2^r (c_0^r - 1) - (c_1^r - 1)^2 = 0 \quad . \quad (21)
\]

The equations (19), (20) and (21) can be easily solved for given values of \( c_0^r, c_1^r \) and \( c_2^r \). We note that the two solutions of (21) for \( \alpha_1^{r+1} \) are equivalent. When one solution results in a set of parameters \( (\alpha_1^{r+1}, \alpha_2^{r+1}, p^{r+1}) \), the values that will be obtained by the second solution will be \( (\alpha_2^{r+1}, \alpha_1^{r+1}, 1 - p^{r+1}) \), and it is clear that the two sets of parameters result in the same distribution for the active periods on the inlets of the switches in the \( r+1 \)-th stage.

Finally, what remains now is calculating \( c_1^r \) and \( c_2^r \) in terms of the parameters \( (\alpha_1^r, \alpha_2^r, \beta^r, p^r) \), describing the arrival process on the arrival paths to a tagged output queue belonging to the \( r \)-th stage. Consider a tagged input link of a switch in the \( r+1 \)-th stage. This, of course, is an outlet of an output queue in a switch belonging to the \( r \)-th stage. Assume now, that on this outlet, the \( k \)-th slot is the first slot of an active period. The probability that the active period will be exactly one slot long, is equal to

\[
c_1^r = \text{Prob} \left[ s_k^r = 1, s_{k+1}^r = 0 | s_k^r > 0, s_{k-1}^r = 0 \right] .
\]

This is equal to the probability that there has been one cell arrival during slot \( k - 1 \) and none during slot \( k \) under the given conditions

\[
c_1^r = \text{Prob} \left[ e_{1,k-1}^r + e_{2,k-1}^r = 1, e_{1,k}^r + e_{2,k}^r = 0 | s_k^r > 0, s_{k-1}^r = 0 \right]
\]

Since, as explained in the previous section, we assume that the number of cell arrivals of type 1 and 2 during consecutive slots on the arrival paths to an output queue form a homogeneous first-order Markov chain, with transition probabilities given by expression (11), the above probability becomes

\[
c_1^r = \frac{1}{i=0} \sum_{i=0}^1 e_{1,k-1}^r (0, 0) \text{Prob} \left[ e_{1,k-1} = i, e_{2,k-1} = 1 - i | s_k^r > 0, s_{k-1}^r = 0 \right] .
\]

Since there cannot have been a cell arrival during slot \( k - 2 \) if \( s_{k-1}^r = 0 \), this yields

\[
c_1^r = \frac{1}{i=0} \sum_{i=0}^1 e_{1,k-1}^r (0, 0) \text{Prob} \left[ e_{1,k-2}^r + e_{2,k-2}^r = 0, e_{1,k-1}^r = i, e_{2,k-1}^r = 1 - i | e_{1,k-1}^r + e_{2,k-1}^r > 0, s_{k-1}^r = 0 \right] .
\]

where we have also used the property that the conditions \( s_k^r > 0 \) and \( s_{k-1}^r = 0 \) are equivalent to \( e_{1,k-1}^r + e_{2,k-1}^r > 0 \) and \( s_{k-1}^r = 0 \). Again, using the property that the numbers of cells arrivals of
both types on the arrival paths is first-order Markov, it is not difficult to show that the above relation can be written as

\[ c_1^r = \frac{1}{1 - e_{00}^r(0,0)} \sum_{i=0}^{1} e_{00}^r(i,1-i)e_{2,1-i}^r(0,0). \]

In a similar way, under the same assumptions, we find for the probability that the active period will be 2 slots long

\[ c_2^r = \text{Prob} \left[ s_k^r = 2, s_{k+1}^r = 1, s_{k+2}^r = 0 | s_k^r > 0, s_{k-1}^r = 0 \right] \]
\[ + \text{Prob} \left[ s_k^r = 1, s_{k+1}^r = 1, s_{k+2}^r = 0 | s_k^r > 0, s_{k-1}^r = 0 \right]. \]

Expressing these probabilities in terms of the arrival process, we find

\[ c_2^r = \sum_{j=1}^{2} \text{Prob} \left[ e_{1, k-1}^r + e_{2, k-1}^r = j, e_{1, k+1}^r + e_{2, k+1}^r = 2 - j, e_{1, k+1}^r + e_{2, k+1}^r = 0 
\right. \]
\[ \left. | s_k^r > 0, s_{k-1}^r = 0 \right]. \]

Using a similar technique as in the calculations concerning \( c_1^r \), the above probability becomes equal to

\[ c_2^r = \sum_{j=1}^{2} \sum_{j=0}^{2-j} \sum_{n=0}^{2-j} e_{i,j-i}^r(n,2-j-n)e_{n,2-j-n}^r(0,0) \]
\[ \text{Prob} \left[ e_{1, k-1}^r = i, e_{2, k-1}^r = j - i | e_{1, k-1}^r + e_{2, k-1}^r > 0, s_{k-1}^r = 0 \right] \]. \hspace{1cm} (22)

Again, using the property that there were no arrivals at all during slot \( k-2 \) on the arrival paths of the output queue, since \( s_{k-1}^r \) must be zero, the above expression finally becomes

\[ c_2^r = \frac{1}{1 - e_{00}^r(0,0)} \sum_{j=1}^{2} \sum_{i=0}^{j} e_{00}^r(i,j-i) \sum_{n=0}^{2-j} e_{i,j-i}^r(n,2-j-n)e_{n,2-j-n}^r(0,0). \] \hspace{1cm} (23)

The quantities \( e_{m,n}^r(k,l) \) occurring in (22) and (23) can be calculated from (7), (9), (11) and (12) for given values of \( (\alpha_1^r, \alpha_2^r, \beta_1^r, \beta_2^r) \). Then, with (22) and (23), the probabilities \( c_1^r \) and \( c_2^r \) are obtained. This allows the calculation of the parameters \( \left( \alpha_1^{r+1}, \alpha_2^{r+1}, \beta_1^{r+1}, \beta_2^{r+1} \right) \), characterizing the arrival process in the \( (r+1) \)-th stage, using (19), (20) and (21).

5 The limiting value of \( \beta^r \)

In this section, we will show that, starting from an initial value \( 0 < \beta^0 < 1 \), the sequence \( \{\beta^r\} \), \( r \geq 1 \), calculated from (15), converges to a limiting value \( \beta_1 \). We first prove the following lemma:

**Lemma**

Assume that a real continuous function \( f(x) \), defined in the whole interval \([0,1]\), satisfies

1. \( 0 < f(0) < f(1) < 1 \). This implies that we can find at least one real number \( x_1 \) \( (0 < x_1 < 1) \) for which \( f(x_1) = x_1 \). Assume that only one such value \( x_1 \) exists.

2. \( f(x) \) increases monotonically for \( 0 < x < 1 \).

Then:
A) If \( x_0 \) satisfies \( 0 < x_0 < x_l \), then \( x_0 < f(x_0) < x_l \). In other words, the sequence \( \{x_0, f(x_0), f(f(x_0)), \ldots\} \) converges to \( x_l \).

PROOF

a) Under the assumption that the equation \( f(x) = x \) has exactly one real root \( x_l \) for \( 0 < x < 1 \), it is clear that \( f(0) - 0 > 0 \) and \( f(x_l) - x_l = 0 \) imply that

\[
f(x_0) > x_0
\]

(24)

for \( 0 < x_0 < x_l \), since \( f(x) \) is assumed to be continuous.

b) Assume that \( f(x_0) > x_l \). Because of \( f(x_l) = x_l \), this would imply \( f(x_0) > f(x_l) \) which is in contradiction with the monotonically increasing property of \( f(x) \). Thus

\[
f(x_0) < x_l
\]

(25)

B) If \( x_0 \) satisfies \( x_l < x_0 < 1 \), then \( x_l < f(x_0) < x_0 \). In other words the sequence \( \{x_0, f(x_0), f(f(x_0)), \ldots\} \) converges to \( x_l \).

PROOF

a) Under the assumption that the equation \( f(x) = x \) has exactly one real root \( x_l \) for \( 0 < x < 1 \), it is clear that \( f(1) - 1 < 0 \) and \( f(x_l) - x_l = 0 \) imply that

\[
f(x_0) < x_0
\]

(26)

for \( 1 > x_0 > x_l \), since \( f(x) \) is assumed to be continuous.

b) Assume that \( f(x_0) < x_l \). Because of \( f(x_l) = x_l \), this would imply \( f(x_0) < f(x_l) \), which is in contradiction with the monotonically increasing property of \( f(x) \). Thus

\[
f(x_0) > x_l
\]

(27)

Summarizing (24), (25), (26) and (27), we see that starting from an initial value of \( x_0 \) between 0 and 1, the sequence \( \{x_0, x_1, x_2, \ldots\} \) with \( x_n \) being defined as \( f(x_{n-1}) \) \( (n \geq 1) \), converges to the limiting value \( x_l \).

What remains now is to show that the right hand side of (15), being a function of \( \beta \)

\[
f(\beta) = \left( \frac{N-1}{N} \left( \frac{N+1}{N} - \frac{2}{N} + \frac{1-N\sigma}{N^2} \beta \right) \right)^N
\]

(28)

satisfies the initial conditions of the above lemma. First of all, it is clear that \( f(\beta) \) is a continuous and real function for \( 0 < \beta < 1 \). Furthermore, since we only consider values of \( N \) and \( \sigma \) that satisfy \( N \geq 2 \) and \( 0 < \sigma < 1 \), one can easily verify that the right hand side of (15) is strictly positive for \( 0 < \beta < 1 \). On the other hand, the condition \( f(\beta) < 1 \) is equivalent to

\[
\beta < \frac{1 + (N-2)\sigma}{1-\sigma}
\]

The right hand side of this inequality is greater than 1, thus this condition is also met for \( 0 < \beta < 1 \). Also, since the denominator of the right hand side of (28) as well as the quantity \((1-\sigma)/N^2\) are strictly positive quantities, \( f(\beta) \) is a monotonically increasing function for positive
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Table 1: \((α₁^r, α₂^r, β^r, p^r)\) for increasing values of \(r\)

values of \(β\). Finally, we must prove that the equation \(f(β) = β\) has only one real positive root in the interval \(0 < β < 1\). The inequality

\[
1 > \frac{N-1}{N} \left( \frac{N+1}{N} - \frac{2α}{N} \right) + \frac{1-β^r}{N} \exp\{2πiθ\}, 0 ≤ θ ≤ 2π
\]

(which is satisfied for all values of \(N\) and \(σ\) discussed above) implies that \(β > |f(β)|\) on the unit circle \(\{β : |β| = 1\}\). Using Rouche's theorem, this implies that the equation \(f(β) = β\) has only one root within the complex unit disk. Since we already know that this equation has at least one real positive solution in this area, this root must be on the positive real axis.

The above analysis proves that both \(β^r\) and \(α^r\) converges to a limiting valued \(β_l\) and \(α_l\), since both parameters satisfy \(\frac{1-β^r}{1-α^r-β^r} = σ\), and \(β^r\) converges to \(β_l\). This, no doubt, is a consequence of the uniform and independent routing of arriving cells in a switching element to the various output queues, a phenomenon that smoothes the correlation in the arrival process. As far as the limiting behavior of \((α₁^r, α₂^r, p^r)\) is concerned, although it appears plausible, there is no formal guarantee that these parameters converge to a limiting value. Nevertheless, numerous numerical examples have shown that this is indeed the case.

## 6 Numerical Results

In Table 1, \((α₁^r, α₂^r, β^r, p^r)\), for increasing values of \(r\), are compared, starting from two different sets of \((α₁^0, β^0, β^1, p^1)\). The first set corresponds with Bernoulli arrivals on the input links of the switches in the first stage, while the second set corresponds with a strongly correlated arrival process. In both cases the load \(σ\) is equal to 0.4 and the number of inlets \(N\) is 16. The consecutive values of \(α\) that would be obtained using a first-order two-state Markov chain for the cell arrivals, are also given.

An important observation is that, whatever the values of the parameters characterizing the correlated arrival process in the first stage, these parameters reach a limiting value after only a few stages. Furthermore, these limiting values are independent of the initial values of the parameters. This seems to imply that these limiting values only depend on \(N\) and \(σ\), the number of inlets of a switch, and the load on each of these inlets.

We define the burstiness factor

\[
K^r \triangleq \frac{1-σ}{1-α^r} = \frac{σ}{1-β^r}
\]
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Table 2: the limiting values of (α_1, β_1, K_1)

as the ratio of the mean length of an active (or, equivalently, passive) period, versus the mean length of an active (passive) period in case of an uncorrelated Bernoulli arrival process on the switch inlets. Thus, K = 1 corresponds to an uncorrelated arrival process, and increasing values of K corresponds to longer active (and passive) periods, and, consequently, increasing burstiness in the arrival process.

To examine the limiting behavior of a 3-state MMBP arrival process versus a 2-state MMBP arrival process (where we have a single on state in the state transition diagram, implying that α''_1 = α''_2 = α'') with the same mean length for the active and passive period. We define

\[
L' \equiv \frac{\frac{\rho \alpha''_1}{(1-\rho)\alpha''_2}}{\frac{\rho \alpha''_1}{(1-\rho)\alpha''_2}}
\]

as the ratio of the variances of the active periods' lengths in the two mentioned cases. The mean lengths of the active and the passive periods will remain the same in each stage, because as became clear from the previous analysis, starting from (α_0, β_0) on the inlets of the first stage, we will find identical values for (α'' and β'') in both cases, regardless of the number of states in the arrival model.

In table 2, the limiting values for α, β, K and L (α_1, β_1, K_1 and L_1) are compared for different values of σ and N. These results reveal that for constant values of σ both K_1 and L_1 increase if N increases. Furthermore, we observe that for constant values of N both K_1 and L_1 increases if σ increases. This means that, the higher the load and the switch size, the more the arrival process on the inlets of a switch in the network will be correlated, under the assumptions of the model.

7 Conclusions

In this paper, we have developed a generic model for characterizing the traffic sources in consecutive stages, based on the assumption that the queueing processes in subsequent network nodes are independent. In this model, the arrival process in each network node was modeled as a superposition of N identical 3-state MMBP processes. We have proved that under these
circumstances, the traffic characteristics converge to a limiting value after a few stages in the network, independent of the characteristics of the traffic sources at the network access point.

References


