MESSAGE WAITING TIMES AND DELAYS IN ATDM SWITCHING ELEMENTS.

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Abstract.

In this paper we study the delay performance of a switching element, to which messages composed of a variable number of fixed-length packets arrive at the rate of one packet per time slot. More specifically, we obtain results in connection with the message waiting times and the message delays. The study is an extension/continuation of previous work, which was mainly concerned with the derivation of buffer occupancies and packet delays.

1. Introduction.

In this paper, we study a switching element which routes incoming packets from a (large) number of sources to a number of destinations. The switch uses a separate "output buffer" for each possible destination. Since packets with different destinations do not compete, it suffices to focus on one (arbitrary) output buffer, to study the delay performance of the switch. We make the assumption that each output buffer operates as an Asynchronous Time Division Multiplexing (ATDM) system, i.e., packets are transmitted from the output buffer asynchronously with respect to their sources, in their order of arrival in the buffer, as long as the latter is nonempty, (see e.g. [1]). Also we assume that: packets are generated by the sources in the form of variable-length entities ("messages" or "sessions") consisting of (a variable) number of packets that belong together. Time is assumed divided into fixed-length intervals, referred to as time slots, such that one slot suffices to transmit exactly one packet from the output buffer; packets are assumed to leave the buffer at the end of a slot. Sources are assumed to deliver their messages to the output buffer at the rate of one packet per slot. The numbers of new messages, generated by the totality of all sources together, during the consecutive slots are modeled as i.i.d. random variables with common probability generating function B(λ). Each message is composed of a random number of packets which, as is customary, is assumed geometrically distributed:

\[ \text{Prob}[\text{message contains } n \text{ packets}] = (1-\sigma)\sigma^{n-1}, \quad n \geq 1. \]

Notice that, in terms of queueing theory, we are confronted here with a queueing system with correlated arrivals, owing to the fact that one message (or session) may cause packet arrivals in the buffer during several consecutive time slots. Although the majority of the pertinent literature assumes uncorrelated arrivals, queueing systems with different forms of dependence in the arrival stream have been considered before; see, e.g., [2-7]. In fact, this study elaborates on the two previous papers [6,7] in which the buffer occupancy, i.e., the number of packets in the buffer, and the packet delay, i.e., the elapsed time between the arrival and the departure of an arbitrary packet in the output buffer were investigated. The aim of the present paper, however, is to derive (closed-form) results for the message waiting time, i.e., the time between the generation of (the first packet of) a message (or session) and the epoch when the transmission of this packet is about to take place, and for the message delay, i.e., the time between the generation of a message and the completion of its transmission from the buffer. To the best of the author's knowledge, no such studies have been reported before. Some results of [6,7] which are needed for a proper understanding of the present paper are summarized in the next section.

2. Preliminary results.

Let us concentrate on one output buffer of the switching element. We define the random variable \( a_k \) as the total number of packets entering the buffer during slot \( k \), \( b_{k+1} \) as the total number of messages generated during slot \( k \), and \( u_k \) as the buffer occupancy, i.e., the total number of packets stored in the buffer, at the beginning of slot \( k+1 \), i.e., just after slot \( k \). In [6,7] it was shown that the following system equations can then be established:

\[ a_{k+1} = b_{k+1} + \sum_{i=1}^{a_k} c_i, \]  

(1)

\[ u_{k+1} = a_{k+1} + (u_k - 1)^+, \]  

(2)

where the \( c_i \)'s are i.i.d. discrete random variables equal to \( 1 \) or \( 0 \) with probabilities \( \sigma \) or \( 1-\sigma \) respectively, and \( x^+ \) denotes the quantity \( \max(0, x) \).

Equations (1) and (2) were used in [6-7] to derive several explicit results in connection with the steady-state packet arrival process and the steady-state occupancy of the output buffer. In particular, the following quantities were derived:
4. Message waiting times.

The waiting time of a message is defined as the time period between the end of the slot during which the message is generated, i.e., the slot during which the first packet of this message enters the output buffer, and the start of the slot during which this first packet will be transmitted. Here it is assumed that packets are transmitted from the buffer in their order of arrival, regardless of the message (or session) they belong to. The message waiting time can be viewed as the time it takes a message to "get through". With the definition given it always consists of an integral number of slots, so that it can be considered as a discrete random variable.

Let $M(\text{arb})$ denote an arbitrary message, which is generated during a slot referred to as slot $J$. Next, let $a^0$ and $b^0$ denote the total number of packet arrivals and the total number of newly generated messages during slot $J$, and $u^0$ the buffer occupancy just after slot $J$. Let $\varphi(i,j,n)$ denote the joint mass function of $a^0$, $b^0$ and $u^0$:

$$\varphi(i,j,n) = \text{Prob}[a^0 = i, b^0 = j, u^0 = n]\ . \tag{11}$$

Notice the difference between the quantities $f(i,j,n)$ and $\varphi(i,j,n)$: the first corresponds to the fraction of slots with $i$ packet arrivals, $j$ new messages generated and a buffer occupancy of $u$ packets just after the slot, whereas the latter describes the fraction of messages which are generated during such a slot, since $M(\text{arb})$ is an arbitrary message. It can be shown, using similar methods as in [7], that $\varphi(i,j,n)$ is given by

$$\varphi(i,j,n) = \frac{i}{B'(1)} f(i,j,n)\ ,$$

proportional to $i$. The joint mass function of $a^0$ and $u^0$ can be easily obtained from this:

$$g(i,n) = \text{Prob}[a^0 = i, u^0 = n] = \sum_{j=0}^{i} \frac{1}{B'(1)} f(i,j,n)\ . \tag{12}$$

The corresponding probability generating function $G(x,z)$ is then given by

$$G(x,z) = E[x^a z^u] = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} x^i z^n g(i,n) = \frac{1}{B'(1)} \frac{\partial E}{\partial z}(x,1,z)\ , \tag{13}$$

where we have used (12) and (9). This result can be further developed by means of the functional equation (8), as follows:

$$G(x,z) = \frac{x z B'(xz)}{B'(1) B'(xz)} F(x,1,z)\ . \tag{14}$$

Now let the random variable $s$ indicate the number of packets entering the buffer during slot $J$, before the first packet of our tagged message $M(\text{arb})$ does. Due to the first-come–first-served queuing rule, the message waiting time $w$ (of $M(\text{arb})$) is then given by

$$w = u^0 - a^0 + s\ . \tag{15}$$
The corresponding probability generating function $W(z)$ can be obtained as

$$W(z) = E[z^{u^0-a^0}] = E\left[ z^{u^0-a^0} E[z^s \mid a^0, u^0] \right].$$

Now, since the first packet of our tagged message $M(\text{arb})$ is just an arbitrary packet with respect to the order of arrival within slot $J$, the random variable $s$ can take the values $0, 1, \ldots, a^0-1$ with equal probabilities, if a total of $a^0$ packets are known to arrive during slot $J$. Therefore, we find

$$E[z^s \mid a^0, u^0] = \frac{1}{a^0} \sum_{\ell=0}^{a^0-1} z^\ell = \frac{z^{a^0-1}}{a^0(z-1)},$$

so that $W(z)$ can be expressed as

$$W(z) = \frac{1}{z-1} \left[ z^{u^0} \frac{1}{a^0} - E\left[ \frac{z^{a^0} - z^{a^0-1}}{a^0(z-1)} \right] \right],$$

or, in view of (13), as

$$W(z) = \frac{1}{z-1} \int \frac{1}{1/z} \frac{G(x,z)}{x} \, dx . \tag{16}$$

Note that, in the last step we have used the identity

$$\frac{G(x,z)}{x} = \frac{\partial}{\partial x} E\left[ \frac{x^{a^0} z^{u^0}}{a^0} \right],$$

which follows easily from (13).

Equation (16) provides us with an explicit expression for the probability generating function of the message waiting times in terms of the function $G$, which, in turn, is given in equation (14) in terms of the function $F$, for which we have derived a functional equation in (8). Although this is not a very transparent way of characterizing the waiting time distribution of the messages, explicit closed-form expressions for the various moments of the message waiting time can be easily obtained from this, because only derivatives of $W(z)$ at $z=1$ are involved there. We illustrate the technique for the case of the mean value $E[w]$ of the message waiting time. $E[w]$ can be obtained from (16) as

$$E[w] = \frac{dW}{dz}(1) = \frac{\partial G}{\partial z}(1,1) - \frac{1}{2} \frac{\partial}{\partial x} G(1,1,1) . \tag{17}$$

The partial derivatives of the $G$-function can be calculated by means of (14) to be

$$\frac{\partial G}{\partial x}(1,1,1) = \frac{B'(1)+B''(1)-B(1)B'\,(1)^2}{B'(1)} \quad \text{and} \quad \frac{\partial G}{\partial z}(1,1,1) = \frac{B'(1)+B''(1)-B(1)B'\,(1)^2}{B'(1)} . \tag{18}$$

Moreover, from the definition of the $F$-function it follows that

$$\frac{\partial F}{\partial x}(1,1,1) = \lambda = \frac{B'(1)}{1-\sigma} , \tag{20}$$

as given in (3), and

$$\frac{\partial F}{\partial z}(1,1,1) = N , \tag{21}$$

as given in (5). Combining (17)–(21), we finally obtain the following explicit formula for $E[w]$:

$$E[w] = N + \frac{(1-\sigma)B''(1) - (2-\sigma)B(1)B'\,(1)^2}{2(1-\sigma)B'(1)} . \tag{22}$$

Higher order moments of the message waiting time can be derived similarly, by evaluating higher order derivatives of $W(z)$. The integral in expression (16) does not represent any difficulties, because the remaining integrals of the integrand and of its derivatives with respect to $x$ in the eventual expressions are equal to zero at $x=1$.

5. Message delays.

We define the delay of a message as the time period between the (end of the) arrival slot of the first packet of this message in the output buffer, and the epoch when the last packet of the message leaves the buffer. The message delay can thus be viewed as the time required to "deliver" the full message to its destination. Just as the message waiting time, the message delay can be considered as a discrete random variable. However, it turns out that a derivation of the whole probability distribution (for instance, in the form of the probability generating function) is virtually impracticable, as it leads to very complicated and unwieldy formulas. Nevertheless, it appears possible to obtain a closed-form expression for the mean message delay. The method is explained below.

Consider again the tagged message $M(\text{arb})$, entering the output buffer during slot $J$. Let $L$ denote the length of $M(\text{arb})$, expressed in packets, and let us indicate the last packet of $M(\text{arb})$ as packet $P$. The message delay $c$ of $M(\text{arb})$ is equal to the time required for the transmission of all those packets that arrive in the output buffer no later than packet $P$ (packet $P$ is included), except for those that have left the buffer by the end of slot $J$. Hence, if we define $a(J+n)$ as the total number of packet arrivals in the buffer during the $n$-th slot following slot $J$ (referred to as slot $\text{arb}^{J+n}$ in the sequel), and $a^* (J+L-1)$ as the total number of packets arriving during slot $J+L-1$, no later than packet $P$ does, the message delay $c$ can be expressed as

$$c = u^0 - a^0 + a^*(J) , \quad \text{if } L = 1 \tag{23}$$

or

$$c = u^0 + \sum_{n=1}^{L-2} a(J+n) + a^*(J+L-1) , \quad \text{if } L \geq 2 , \tag{24}$$

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where the summation is equal to zero for \( L = 2 \), by convention.

The mean message delay \( E[c] \) is given by

\[
E[c] = \sum_{\ell = 1}^{\infty} (1-\sigma)^{\ell-1} E[c|L=\ell].
\]  

(25)

In order to obtain \( E[c|L=\ell] \) it thus remains for us to find expressions for the quantities \( E[c|L=\ell] \). This can be done as explained below.

**The case \( L = 1 \):**

From equation (23) on the one hand, and equation (15) from the previous section, on the other hand, it is easily seen that

\[
c = w + 1, \quad \text{if } L = 1. \]

It follows that

\[
E[c|L=1] = E[w] + 1
\]

\[
= N + 1 + \frac{(1-\sigma)B''(1) - (2-\sigma)B'(1)^2}{2(1-\sigma)B'(1)}. \quad (26)
\]

**The case \( L \geq 2 \):**

Suppose that \( M(\text{arb}) \) contains \( \ell \geq 2 \) packets, i.e., \( L = \ell \). From equation (24) we then have

\[
E[c|L=\ell] = E[u^0] + \sum_{n=1}^{\ell-2} E[a(J+n)] + E[\alpha(J+\ell-1)] . \quad (27)
\]

The quantities in the right hand side of this equation can be derived as follows.

First, according to (13), \( E[u^0] \) is nothing else than

\[
E[u^0] = \frac{\partial G}{\partial x}(1,1),
\]

which is given by equations (19) and (21); we get:

\[
E[u^0] = N + \frac{B'(1)+B''(1)-B'(1)^2}{B'(1)}. \quad (28)
\]

Next, in order to determine the sum in (27), we note that, in view of system equation (1), \( a(J+n) \) can be expressed (for all \( n \) between 1 and \( \ell-1 \)) as

\[
a(J+n-1),
\]

\[
a(J+n) = b(J+n) + 1 + \sum_{i=1}^{n} c(i,n), \quad (29)
\]

where \( b(J+n) \) denotes the number of new messages generated during slot \( J+n \), the term + 1 accounts for the packet of \( M(\text{arb}) \) which enters the buffer during this slot, and the sum represents the packets entering the buffer during slot \( J+n \) belonging to messages, other than \( M(\text{arb}) \), whose arrival was already in progress one slot earlier; the \( c(i,n) \)'s are i.i.d. random variables equal to 1 or 0 with probability \( \sigma \) or \( 1-\sigma \) respectively.

Taking expected values, we obtain from (29)

\[
E[a(J+n)] = B'(1) + 1 + \sigma \left\{ E[a(J+n-1)] - 1 \right\},
\]

and further, by recursively applying this result,

\[
E[a(J+n)] = \frac{B'(1) + 1 - \sigma}{1-\sigma} \frac{1-\sigma^n}{1-\sigma} + E[a(J)] \sigma^n .
\]

Here \( E[a(J)] \), of course, is identical to \( E[a^0] \), which can be derived from (13) as

\[
E[a(J)] = \frac{\partial G}{\partial x}(1,1),
\]

or, using (18) and (20),

\[
E[a(J)] = \frac{B'(1) + B'(1) + B''(1) - B'(1)^2}{B'(1)}.
\]

We thus find

\[
E[a(J+n)] = 1 + \frac{B'(1) + B''(1) - B'(1)^2}{B'(1)} \sigma^n . \quad (30)
\]

The last term in equation (27) can be shown to be given by

\[
E[\alpha(J+\ell-1)] = E[a(J+\ell-1)] + 1, \quad (27)
\]

in view of the fact that packet \( P \) can be any of the \( a(J+\ell-1) \) arrivals during slot \( J+\ell-1 \) with equal probability. Using (30) we thus get

\[
E[a(J+\ell-1)] = 1 + \frac{B'(1) + B''(1) - B'(1)^2}{B'(1)} \sigma^{\ell-1} . \quad (31)
\]

Substitution of (28), (30) and (31) in (27) now leads to an expression for \( E[c|L=\ell] \), which, in turn, can be used in (25), along with (26), to derive the mean message delay; the final result is

\[
E[c] = N + 1 + \frac{(1-\sigma)B''(1) + 2(2-1-\sigma)B'(1)^2}{2(1-\sigma)^2 B'(1)} + \frac{\sigma}{1-\sigma} . \quad (32)
\]

A comparison of equations (6) and (32) shows that there is a very simple relationship between the mean packet delay \( E[d] \), obtained by a different analytic approach in [7], and the mean message delay \( E[c] \):

\[
E[c] = E[d] + \frac{\sigma}{1-\sigma} = E[d] + \frac{1}{1-\sigma} - 1 .
\]
In words: the mean message delay is equal to the sum of the mean packet delay and the mean message length \(1/(1-\sigma)\), diminished by \(1\). It can be shown that this is due to the geometric nature of the message length distribution.

6. Illustration of the results.

To conclude, let us consider a numerical example. Suppose the messages are generated according to a Poisson distribution with mean \(q\), i.e.,

\[ B(x) = e^{q(x-1)} , \]

and that the mean message length is 100, i.e., \(\sigma = 0.99\). According to (3), the packet arrival rate per slot is then equal to

\[ \lambda = 100 q . \]

Figure 1 shows plots of the mean packet delay \(E_d\), the mean message waiting time \(E[w]\) and the mean message delay \(E[c]\), versus \(\lambda\), for this particular case. The curves make clear that

\[ E[w] \leq E[d] \leq E[c] , \]

which is also generally true, for any message generation distribution.

References.


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