A note on the discretization of Little’s result

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Abstract

By considering discrete-time queueing systems as special cases of continuous-time queueing systems, we derive a discrete-time equivalent of Little’s result. Our result is general in the sense that no assumptions are made regarding the exact details of arrival and departure processes within a discrete-time unit.

1. Introduction

Little’s result [4,7] relates the mean queue contents as seen by an outside observer—i.e., at a random point in time—to the mean waiting-time experienced by a random customer. Applicability of Little’s result includes both single network nodes—including statistical multiplexers, traffic shapers, switching modules and rate adapters—and complete networks.

The adoption of the asynchronous transfer mode (ATM) [3] for modern digital communication networks, has led to an increased interest in the particular class of queueing systems where time is slotted. For these ‘discrete-time’ queueing systems, it is common practice to observe queue contents at equally spaced random slot boundaries—each slot represents one time-unit—and to express customer waiting-time in (number of) slots rather than in absolute time [1,2,6]. These quantities are known to be similarly related as their continuous-time counterparts for particular discrete-time queueing systems (see e.g. [1,2]) and in the framework of a stationary discrete-time G/G/m queueing system [5].

The present contribution relates the discrete-time equivalent of Little’s result to its continuous-time counterpart in both the deterministic and stationary framework [7]. The presented approach was previously used to derive an equivalent relation for discrete-time GI–G-1 queueing systems [1].

2. Discretization of Little’s result

Discrete-time queueing analysis concerns mean queue lengths—either as mean value in stationary regime or as time-average—observed at random slot boundaries. These discrete observation epochs imply that this quantity does not depend on the exact arrival and departure times within slots. Let EN denote the mean queue contents at random slot boundaries for the system under consideration, then EN also denotes the mean queue contents for an alternative system where all arrivals and departures are rescheduled to (just before) slot boundaries. In this new system, the queue contents only changes at slot boundaries, and therefore EN also denotes the mean queue contents at random points in time for the system with rescheduling.

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Comparison of the waiting-time in the original system and the system where arrivals and departures are rescheduled then yields,

$$E_W = EW^* - ET_a + ET_d,$$

(1)

where $E_W$ denotes the mean customer waiting-time in the system under consideration, where $EW^*$ denotes the mean customer waiting-time in the system with rescheduling and where $ET_a$ and $ET_d$ denote the mean arrival time and the mean departure time of a random customer relative to the preceding slot boundary. Clearly, Little’s result applies to the system with rescheduling, and therefore

$$EN = \lambda (EW + ET_a - ET_d),$$

(2)

where $\lambda$ denotes the arrival intensity.

Although Eq. (2) does relate mean customer waiting-time and mean queue contents at random slot boundaries, direct use is rather limited due to the presence of $ET_a$ and $ET_d$.

2.1. Synchronization to slot boundaries

Numerous discrete-time analyses assume that service of customers is synchronized to slot boundaries implying a.o. that customers cannot start service before the beginning of the slot following their arrival slot. In addition, service of these customers takes an integer number of slots. Therefore, customers always leave the system at (just before) slot boundaries, i.e., $ET_d = \Delta$, where $\Delta$ denotes the slot length.

The mean customer waiting-time in this case can be decomposed in a mean synchronization delay ($\Delta - ET_a$) and a mean queueing delay $EW_d$ expressed in number of slots,

$$EW = \Delta - ET_a + EW_d \Delta.$$  

(3)

Let $\lambda_d = \lambda \Delta$ denote the mean number of arrivals per slot, substitution of the former expression into Eq. (2) then yields

$$EN = \lambda_d EW_d.$$  

(4)

The former expression is clearly similar to Little’s result.

2.2. Early and late arrivals

Systems with early and late arrivals offer an alternative for the previous methodology. Arrivals occur either at the beginning (early) or at the end (late) of the consecutive slots and service takes an integer number of slots. For both early and late arrivals, departures occur before slot boundaries (this implies that the observer will not notice early arriving customers with a single slot service time). Let $f$ denote the fraction of early arrivals, then, Eq. (2) yields

$$EN = \lambda_d (EW_d - f),$$  

(5)

where $\lambda_d$ denotes the mean number of arrivals per slot and where $EW_d$ denotes the mean customer waiting-time expressed in number of slots.

3. Conclusions

We have applied Little’s result to discrete-time queueing systems and shown that for a large class of queueing systems (synchronized systems and systems with late arrivals), mean queue contents at random slot boundaries and mean customer delay (in slots) are similarly related as their continuous-time counterparts.

References