where a high performance asynchronous pipeline control method is required. When using dynamic logic, the technique achieves a performance improvement of 15% over an optimal two-phase design, and 36% over an optimal four-phase design.

Acknowledgment: We gratefully acknowledge the support of the Australian Research Council, and thank A. Beaumont-Smith for helpful suggestions.

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Electronics Letters Online No: 1996/398

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References


Variance of buffer contents and delay in ATM queues

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Indexing terms: Asynchronous transfer mode, Queueing theory

A discrete-time infinite-capacity queue with a finite number of independent heterogeneous traffic sources is studied. The sources are modelled as on/off-sources with geometric on-periods and off-periods having a general distribution. The authors present a technique to derive closed-form expressions for the variance of the buffer contents and the delay.

Introduction: Exact closed-form formulas for the variances of both buffer contents and delay have been derived in [2], for a discrete-time single-server infinite-capacity queue with heterogeneous on/off-sources, under the assumption of geometric distributions for both the on-periods and the off-periods. In this Letter, we extend the results to the case of a general on-period distribution. This allows us to investigate the effect of the precise form of this distribution on the buffer behaviour.

Model: We consider a discrete-time single-server queue with infinite buffer capacity. The time axis is divided into fixed-length slots, where one slot is the time interval to transmit exactly one ATM cell. The buffer is fed by N independent, not necessarily identical bursty sources. There are T traffic types, and for traffic type i, 1 ≤ i ≤ T, there are Xi sources. Each source alternates between on-periods, during which it generates one cell per slot, and off-periods, during which no cells are generated. We assume in this Letter that the lengths of the on-periods and the off-periods are independent of each other. For a source of type i, the off-periods are geometrically distributed with parameter βi, whereas the on-periods are i.i.d. with arbitrary probability generating function (PGF) Ai(z) and probability mass function αi(z).

The load σi of the source is then given by σi = (1 − βi)Ai(1)(1 − Ai(1)). We define the ‘burstiness factor’ K as the ratio of the mean length of an on-period (or an off-period) in our model, to the corresponding quantity in case of a Bernoulli arrival process with the same load σi, i.e. K = (1 − σi)Ai(1)/σiAi(1) + (1 − βi)Ai(1). Also, we define the ‘variance factor’ L as the ratio of the variance of the on-period length in our model, to the variance of a geometrically distributed on-period with the same mean length, i.e. L = E[(U − A1(1) − A1(1))/A1(1)]/(A1(1) − 1). Finally, we denote by M, the third moment of the on-period distribution.

Analysis method: As mentioned before, each source will alternate be passive (state 0), or active. An active source is called in state i, i ≥ 1, if it is in the ith slot of an on-period. Hence, each source can be characterised by an infinite-dimensional Markov chain with states i, i ≥ 0, and transition probabilities as shown in Fig. 1, where p(i,j) is the probability that an on-period consists of at least j + 1 slots, given that it consists of at least i slots, i.e.

\[ p(i,j) = \sum_{n=0}^{\infty} a_n(i) \left( \sum_{m=0}^{\infty} a_m(n) \right)^{-1} \]

(1)

Now let d(i) be the number of sources of type i in state i during an arbitral slot in the steady state, and let s represent the system contents (i.e. the number of cells in the buffer, including the potential cell in transmission) at the beginning of the next slot. In [3], the following fundamental functional equation has been obtained for the joint PGF P_{X_1, ..., X_T, Z}(d, s, z) of d(i), i ≥ 1, 1 ≤ s ≤ T and z:

\[ zP_{X_1, ..., X_T, Z}(d, s, z) = \prod_{i=1}^{T} [\beta_i + (1 - \beta_i)zX_{1,i,1}]^{d(i)} \times \{P_{H_1(X_1, z), ..., H_T(X_T, z), z}[(s-1)(1-p)] \}

(2)

Here \( p = \sum_{n=1}^{\infty} n \sigma_i \) is the aggregate input load, \( x_i = (X_{i,1}, X_{i,2}, ...) \)

and

\[ H_i(x_{1,i}, z) = \frac{1 - p_i(1) + p_i(1)x_{2,i,z}}{\beta_i + (1 - \beta_i)zX_{1,i,1}} \times \frac{1 - p_i(2) + p_i(2)x_{3,i,z}}{\beta_i + (1 - \beta_i)zX_{1,i,1}} \times \ldots 

(3)

Furthermore, the PGF P_{X_1, ..., X_T, 1}(d, s, z) of the cell arrival process has been derived in [3] as

\[ P_{X_1, ..., X_T, 1}(d, s, z) = \prod_{i=1}^{T} (1 - (1 - \sigma_i)^{d(i)}) \left( \frac{(1 - \beta_i) \sum_{n=0}^{\infty} a_m(n)z^{d(i)}}{1} \right) \]

(4)

Although it seems very hard to solve eqn. 2 for the \( P \)-function explicitly, in general, closed-form expressions can be obtained from eqn. 2 for all the moments of the system contents. To do this, first express the desired moment in terms of the consecutive partial derivatives (PDs) of the function \( P_{X_1, ..., Z, 1} \) with respect to \( z \), for \( x_i = 1, 1 ≤ i ≤ T \) and \( z = 1 \), and secondly, to evaluate these PDs. The simplest method to accomplish the latter task is as follows. First, consider only those values for \( x_i \) and \( z \), for which the arguments of the \( P \)-functions in both sides of eqns. 2 are equal, i.e. for which \( x_i = H_i(x_{1,i}, z), 1 ≤ i ≤ T \). It is worth noting that in the resulting equations the various traffic types (i.e. values of \( n \)) are strictly noninterfering, which facilitates the analysis considerably. However, solving them for \( x_i \) in terms of \( z \), we may get more than one set of solutions. Only one of these sets, which we will denote by \( \chi(z) = (\chi_1(z), \chi_2(z), ...) \) has the additional property that \( x_i = 1 \) for \( z = 1 \). Choosing \( \chi_i = \chi_i(z) (1 ≤ i ≤ T) \) in eqn. 2 then leads to

\[ P_{X_1, ..., Z, 1}(d, s, z) = (1 - p \sum_{i=1}^{T} (1 - \beta_i)\chi_i(z)z)^{d(i)} \]

(5)
Next, evaluation of the consecutive derivatives of eqn. 4 with respect to $z$ for $z = 1$ gives the PDFs of $P_{x_{i-1}, ..., x_{i-r}, z}$ with respect to $z$ in terms of the one hand, PDFs with respect to $x_{i}$, and on the other hand mixed PDFs with respect to $x_{i}$ and $z$. The former PDFs are easily calculated from eqn. 3, whereas the latter PDFs can be eliminated from the results by means of the original functional eqn. 2.

Under the assumption of a first-come first-served queuing discipline, all the moments of the cell delay $d$ (i.e. the number of slots between the end of a cell’s arrival slot and the end of the slot during which the cell is transmitted) can be expressed in terms of the corresponding moments of $s$ by means of the general relationship between system contents and delay reported in [1].

**Results:** We will now give the results obtained with the technique described above. The mean buffer contents is given by

$$E[s] = p\frac{1}{2(1-p)} \sum_{i=1}^{T} N_i \sigma_i(p-\sigma_i)[K_i + L_i(K_i-1)+\sigma_i(L_i-1)]$$

(6)

The variance of the system contents can be derived as

$$\text{var}[s] = \frac{1}{(1-p)} \sum_{i=1}^{T} N_i(L_i+1)(K_i+\sigma_i-1) \times \sigma_i[(p-\sigma_i)(3\sigma_i^2 - 6\sigma_i + 3 + 2p\sigma_i) + \sigma_i(1-p\sigma_i)]$$

$$+ \frac{1}{4(1-p)} \sum_{i=1}^{T} N_i(L_i+1)^2(K_i+\sigma_i-1)^2 \sigma_i^2$$

$$+ \frac{1}{3(1-p)} \sum_{i=1}^{T} N_i \sigma_i[(p-\sigma_i)(-12\sigma_i^3 + 14 + 3p\sigma_i) + \sigma_i(1+p\sigma_i)]$$

$$+ \frac{1}{3(1-p)} \sum_{i=1}^{T} N_i M_i \tau_i(1-\sigma_i)^2(\sigma_i - 2)$$

$$- \frac{1}{4} \sum_{i=1}^{T} N_i \sigma_i[(L_i+1)(K_i+\sigma_i-1)-2\sigma_i]$$

$$+ \frac{1}{2(1-p)} \sum_{i=1}^{T} N_i \sigma_i[(1-\sigma_i)$$

$$\times ((L_i+1)(K_i+\sigma_i-1)-2\sigma_i + p-\sigma_i)]$$

$$-E[s] \sum_{i=1}^{T} N_i \sigma_i[(L_i+1)(K_i+\sigma_i-1)-2\sigma_i] + E[s] - (E[s])^2$$

(7)

whereas the cell-delay variance is given by

$$\text{var}[d] = \frac{\text{var}[s]}{p} - (1-p) \left(\frac{E[s]}{p}\right)^2$$

(8)

**Discussion:** The above general results were found to be in agreement with those obtained in [2], for geometric on-periods, i.e. for $L_i = 1$, $1 \leq t \leq T$. As an illustration of our results, Fig. 2 shows the cell-delay variance for $N = 10$ homogeneous sources with given off-period distribution and various on-period distributions with given mean but different variances. From this Figure, we observe that the delay variance is an increasing function of the on-period variance. We may conclude that more variable on-period lengths give rise to larger and more variable queue lengths and delays.

**Acknowledgment:** The authors wish to thank the Belgian National Fund for Scientific Research (NFWO) for support of this research.

© IEE 1996
Electronics Letters Online No: 19961341
8 August 1996
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**References**

**Relation between OFDDs and FPRMs**

R. Drechsler and B. Becker

*Indexing terms:* Boolean functions, Reed-Muller codes

Ordered functional decision diagrams (OFDDs) are often used as a data structure for representing fixed polarity Reed-Muller expressions (FPRMs). The authors show that OFDDs are never much larger in size than FPRMs. In contrast there exists Boolean functions for which FPRMs always have exponential size, while the OFDD representation remains small. In this sense the authors show that OFDDs are a good data structure for FPRM representation.

**Introduction:** OFDDs are used for FPRM representation in many applications (see e.g. [1, 2, 4, 5]). Therefore, we study the relationship between the two forms of representation of Boolean functions.

Essential notation and definitions are presented as follows, which are important for the understanding of the Letter.

A fixed polarity Reed-Muller expression (FPRM) is an exclusive-OR of AND product terms, where each variable only appears in complemented or uncomplemented form, but not both. FPRMs are a canonical representation of Boolean functions $f : B^n \rightarrow B$, if the polarity for each variable is fixed. The size of an FPRM is given by the number of terms.

The choice of the polarity largely influences the size of the resulting FPRM, as shown by the following example:

**Example 1:** Let $f : B^n \rightarrow B$, given by

$$f = x_1 x_2 \ldots x_n$$

Thus, only one term is needed, if all variables are complemented. If all variables are uncomplemented the resulting expression consists of $2^n$ terms, i.e.

$$f = 1 \oplus x_1 \oplus x_2 \oplus \ldots \oplus x_1 x_2 \ldots x_n$$

**Fig. 2 Cell-delay variance against total load, for N = 40, T = 1, N(t) = 16**

(i) mixture of 2 geometrics ($L = 2.5$, $M = 9.7314$)
(ii) geometric ($L = 1$)
(iii) negative binomial ($L = 0.53125$)
(iv) constant ($L = 0$)