An approximate analytical technique for the performance evaluation of ATM switching elements with burst routing

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Accepted 28 December 1994

Abstract

In this paper we consider an ATM (Asynchronous Transfer Mode) switching element with output queuing. One separate infinite-capacity output buffer is used for each possible destination (output). The cell arrival processes on the inlets of the switching element are of a bursty nature. Specifically, geometrically distributed on/off-periods are assumed to describe this burstiness. Cells are routed from the inputs to the outputs of the switching element in a uniform but correlated manner, i.e., all cells belonging to the same on-period are routed to the same destination, but all destinations are equiprobable.

The queueing performance of the switch is analyzed here by a combination of analytic techniques and approximations. Close analytic upper bounds are obtained for such measures as the means and the tail distributions of the buffer contents and the cell delay in a tagged output buffer. The formulas are easy to evaluate and the results are useful in practice, for instance, to calculate the cell loss ratio or the delay jitter in an output queue. The influence of the degree of burstiness in the cell arrival processes is investigated. Also, the results are compared with the case of uncorrelated routing.

Keywords: ATM switching elements; Performance evaluation; Discrete-time queues; Bursty arrivals; Correlated routing

1. Introduction

Although discrete-time queueing analysis has been applied (occasionally) since over twenty years in the performance evaluation of various types of slotted communication systems (see, e.g., [1–7]), the conception of ATM-based multiservice networks seems to have caused an increased interest in discrete-time models since the end of the 1980’s. In particular, discrete-time approaches have been adopted on
various occasions for the queueing analysis of (ATM) statistical multiplexers and (ATM) switching elements. In both these applications, a finite number of random traffic sources deliver data units ("cells") to a common buffer, from which cells are transmitted at the rate of one per slot as long as the buffer is nonempty. In case of a multiplexer, all cells have the same (common) destination and the cell sources are directly connected to the common buffer via the inlets of the multiplexer. In case of switching elements, however, a routing mechanism (between the sources and various destinations) determines the actual arrival process of cells in the buffer (corresponding to one of many destinations). A queueing model for a multiplexer thus essentially implies a statistical description of the traffic sources that generate the cells to be transmitted. Studying the queueing performance of a switching element, however, also requires an (additional) statistical description of the applied routing mechanism, and is therefore, in general, more complicated.

Performance studies of multiplexers and switches can be categorized according to the nature of the sources, i.e., the cell arrival processes on the inlets, and, in the case of switching elements, according to the nature of the routing mechanism. The simplest models assume uncorrelated arrival streams on the inlets: examples include [1,3,4,8] for multiplexers and [9–15] for switching elements (with independent routing mechanism). However, in view of the rather complicated traffic patterns which may occur in multiservice networks, several researchers have also concerned themselves with more general, i.e., "nonindependent" or "bursty" arrival models, during the last several years. Although other types of characterizations have been used as well, source models of the on/off-type have been particularly popular, both in the continuous-time (see, e.g., [16]) and the discrete-time domain [6,17–24]. Concentrating on the discrete-time studies, we note that [6,17–21] deal with multiplexers, whereas [22–24] have to do with switching elements. In [6,17–20] both the on-periods and the off-periods of the traffic sources are modeled as geometrically distributed random variables. A somewhat deviating source model is considered in [21], where the on-periods are modeled as geometrically distributed multiples of a given fixed number of slots. The same source models are also found in switching element analyses: geometric on/off-periods in [22–23] and geometric multiples of a fixed interval for the on-periods in [24].

As to the statistical description of the routing process, most studies where the arrival processes on the inlets of the switch are modeled as uncorrelated, such as [9–15], have also assumed uncorrelated routing mechanisms (no correlation between the destinations of consecutive cell arrivals on any given inlet). Among the studies with on/off-type input processes, independent uniform routing of cells was assumed in [22], whereas in [23–24] various types of correlation in the routing process were also considered. However, to the best of the authors' knowledge, a full queueing analysis has never been reported for sources with geometric on/off-periods in case of "fully correlated routing" or "burst routing", i.e., in case that all the cells arriving during the same on-period are destined for the same output, which may describe a worst-case situation in early stages of the network. The purpose of the present paper is exactly to provide this kind of analysis. The mathematical model considered here might be criticised as simpler than it ought to be. However we believe this paper provides a reasonable compromise between heavy mathematics and realistic modeling. Future work should consider more refined models of routing correlation. In order to analyze the system, we have adapted an analytic method, developed in [24], to tackle the problem, which is basically a combination of a generating-functions approach with bounding (approximation) techniques. As a result, approximate analytic formulas are obtained for various quantities of interest. Specifically, the analysis yields close upper bounds for such measures as the means and the tail probabilities of the buffer contents and the cell delay in a tagged output queue of the switching element.

The outline of the paper is as follows. In Section 2, we describe the switching element under study and state the main elements of the mathematical model. In Section 3, we establish a fundamental functional equation in terms of a trivariate generating function which characterizes the queueing behavior of the tagged output buffer. Section 4 concentrates on the steady-state cell arrival process in the tagged output
buffer and introduces bounds for some remaining unknowns of the analysis. These approximations are then used in Sections 5 and 6 to derive close upper bounds for the mean and the tail probabilities of the buffer occupancy. In Section 7, corresponding bounds are derived for the cell delay. Section 8 is concerned with verifying the accuracy of the analytic approximations. Finally, in Section 9, we present a number of numerical examples and investigate the influence of the burstiness of the sources and the correlation in the routing process on the performance of the switch.

2. Mathematical model

In this paper we consider a symmetric switching element for ATM cells, with $N$ inlets and $N$ outlets. Cells enter the switch via one of the inlets and are then routed to one of the outlets (according to their destination address) where they are temporarily buffered in a designated output queue (output buffer) to await the transmission of earlier cell arrivals with the same destination. Each output queue is assumed to have an unlimited storage capacity for ATM cells. Synchronous transmission is used on both the input links and the output links of the switch, i.e., time is divided into fixed-length slots and at most one ATM cell is transmitted during each slot on any of these links. Note that due to the slotted transmission mode on the outlets of the switch, a cell can never leave the output buffer before the end of the slot right after its arrival slot.

We assume that cell arrivals on the inlets of the switching element are generated by $N$ independent sources with identical statistical characteristics. The sources are of a bursty nature, which will be described by means of an on/off-type of model. Specifically, we assume that each source stochastically alternates between an active state ("on") and a passive state ("off"). When active, a source generates exactly one ATM cell per slot; when passive, a source does not generate any ATM cells at all. The lengths of the active and passive periods of a source are modeled here as independent, geometrically distributed random variables, with parameters $\alpha$ and $\beta$ respectively, i.e.,

$$\Pr[\text{active period} = n \text{ slots}] = (1 - \alpha)\alpha^{n-1}, \quad n \geq 1;$$

$$\Pr[\text{passive period} = n \text{ slots}] = (1 - \beta)\beta^{n-1}, \quad n \geq 1.$$

This implies that the sources are correlated in a first-order Markovian way: the probability that any given source is active or passive in any given slot is fully determined by the state of this source in the previous slot. Specifically, if $p(Y \mid X)$ denotes the probability that a source is currently in state $Y$, given that it was in state $X$ in the preceding slot, we have

$$p(\text{active} \mid \text{active}) = \alpha; \quad p(\text{passive} \mid \text{active}) = 1 - \alpha;$$

$$p(\text{active} \mid \text{passive}) = 1 - \beta; \quad p(\text{passive} \mid \text{passive}) = \beta.$$

Note that the classical Bernoulli arrival model is obtained here as a special case, i.e., for $\alpha + \beta = 1$.

The average load of one source is defined as the fraction of time (slots) this source spends in the active state, and is thus given by

$$p = \frac{E[\text{active period}]}{E[\text{active period}] + E[\text{passive period}]} = \frac{1}{1 - \alpha} \left( \frac{\frac{1}{1 - \alpha}}{\frac{1}{1 - \alpha} + \frac{1}{1 - \beta}} \right),$$
where $E[\ldots]$ denotes the expected value of the expression between square brackets. This implies that, in general, the mean lengths of active and passive periods can be expressed as

$$E[\text{active period}] = \frac{1}{1-\alpha} = \frac{K}{1-p} \quad \text{and} \quad E[\text{passive period}] = \frac{1}{1-\beta} = \frac{K}{p},$$

for some value of the real quantity $K$. It is clear that the statistical properties of a source can be fully characterized by the parameters $p$ and $K$ (instead of $\alpha$ and $\beta$): the load $p$ is a measure for the ratio of the active and passive periods, whereas the constant $K$ (in the sequel referred to as the "burstiness factor") is representative for the absolute lengths of these periods. High values of $K$ are indicative of a high degree of correlation in the cell arrival process. A classical (uncorrelated) Bernoulli arrival process (with load $p$) corresponds to $K = 1$.

As mentioned before, we assume in this paper that the destination addresses of consecutive cell arrivals on a given inlet are not independent. Specifically, we assume that all the ATM cells generated by a given source during the same active period belong together and have exactly the same destination. Cells generated during different active periods (of a given source), however, are routed entirely independently. In addition, all the destination addresses are equiprobable (in the long run). Stated otherwise, we are considering in this paper uniform and independent routing of bursts (i.e., active periods) rather than (individual) cells.

We observe that the (eventual) cell arrival process in a selected ("tagged") output queue is determined by the interaction between source characteristics and routing mechanism. Specifically, in each slot, each of the inlets of the switching element, from the point of view of the tagged output queue, may be either "active" (if it delivers a cell to the tagged output queue) or "blocked" (if it does not). An inlet is active (state A) if it receives a cell from the corresponding source and this cell is routed to the tagged output. An inlet may be blocked (B) for several reasons: either it does not receive any cells from the associated source (state B1), or it does receive a cell from the source, but this cell is not routed to the tagged output (state B2). We therefore conclude that, from the point of view of the tagged output queue, each inlet can be characterized by a three-state Markov chain with states A, B1 and B2, and transition probabilities as indicated in Fig. 1.

3. Fundamental relationships and functional equation

Let us define the random variables $e_k$ and $v_k$ as the number of cell arrivals in the tagged output queue and the non-tagged output queues respectively, i.e., the number of inlets in state A and B2.

![Markov chain model of an inlet.](image-url)
respectively, during slot \( k \). Then, in view of Fig. 1, \( e_k \) and \( v_k \) can be derived from \( e_{k-1} \) and \( v_{k-1} \) as follows:

\[
e_k = \sum_{i=1}^{c_{k-1}} c_i + \sum_{i=1}^{N-c_{k-1}-v_{k-1}} d_i \tag{1}
\]

and

\[
v_k = \sum_{i=1}^{c_{k-1}} c'_i + \sum_{i=1}^{N-c_{k-1}-v_{k-1}} f_i. \tag{2}
\]

The reasoning behind Eq. (1) is that \( e_k \) contains one unity for each inlet which was active during slot \( k-1 \) and which remains in state A during slot \( k \), on the one hand, and one unity for each inlet which was in state B during slot \( k-1 \) and which changes to state A in slot \( k \), on the other hand. The random variable \( c_i \) in (1) takes on the values 0 or 1, and equals 1 if and only if the \( i \)th active inlet during slot \( k-1 \), remains in state A during the next slot, what happens with probability \( \alpha \), due to the geometric distribution of the active periods of a source. A similar reasoning holds for the random variable \( c'_i \) in (1) and for Eq. (2). Hence, we have that the \( c_i \)'s and the \( c'_i \)'s are i.i.d. random variables with common probability generating function (pgf)

\[
C(z) = E[z^{c_i}] = E[z^{c'_i}] = 1 - \alpha + \alpha z, \tag{3}
\]

whereas the pairs \((d_i, f_i)\) are i.i.d. with joint pgf

\[
Q(x, y) = E[x^{d_i}y^{f_i}] = \beta + \frac{1 - \beta}{N} x + \frac{(1 - \beta)(N - 1)}{N} y. \tag{4}
\]

Moreover, the \( c_i \)'s and the \( c'_i \)'s and the pairs \((d_i, f_i)\) are mutually independent (because they refer to different inlets of the switching element).

Let \( s_k \) be the random variable representing the number of cells stored in the tagged output queue at the beginning of slot \( k \), i.e., just after slot \((k-1)\). Then the evolution of the buffer occupancy is described by the following equation:

\[
s_{k+1} = (s_k - 1)^+ + e_k, \tag{5}
\]

where \((.)^+\) denotes \( \max(0,.) \). Equations (1), (2) and (5) imply that the set \( \{(e_{k-1}, v_{k-1}, s_k)\} \) is a Markov chain. Let us define the three-dimensional joint pgf of \( (e_{k-1}, v_{k-1}, s_k) \) as

\[
P_k(x, y, z) = E[x^{e_k}y^{v_k}z^{s_k}].
\]

From (5) it then follows that

\[
P_{k+1}(x, y, z) = E[x^{e_k}y^{v_k}z^{s_k+1}] = E[(x^{e_k})y^{v_k}z^{s_k+1}].
\]

Next, using (1) and (2) and averaging over the distributions of the \( c_i \)'s, the \( c'_i \)'s and the \((d_i, f_i)\)'s yields

\[
P_{k+1}(x, y, z) = \left[Q(xz, y)\right]^N E\left[\left(\frac{C(xz)}{Q(xz, y)}\right)^{e_k-1} \left(\frac{C(y)}{Q(xz, y)}\right)^{v_k-1} z^{s_k+1}\right],
\]

where the expectation is over the joint distribution of \( (e_{k-1}, v_{k-1}, s_k) \). As \( s_k = 0 \) implies that \( e_{k-1} = 0 \), it follows that

\[
P_{k+1}(x, y, z) = \left[Q(xz, y)\right]^N \left\{ \frac{1}{z} P_k \left(\frac{C(xz)}{Q(xz, y)}\right)^{e_k-1} \left(\frac{C(y)}{Q(xz, y)}\right)^{v_k-1} z^{s_k+1}\right\}
\]

\[
+ \left(1 - \frac{1}{z}\right) \Pr[s_k = 0] \sum_{j=0}^{N} \Pr[v_{k-1} = j | s_k = 0] \left(\frac{C(y)}{Q(xz, y)}\right)^j.
\]
We now assume that the average number of cell arrivals in the tagged output queue is strictly less than 1, so that the queueing system can reach a steady state. In this case
\[
\lim_{k \to \infty} P_k(x, y, z) = \lim_{k \to \infty} P_k(x, y, z) = P(x, y, z),
\]
where \( P(x, y, z) \) is the steady-state joint pgf. Therefore, \( P(x, y, z) \) must satisfy the following functional equation:
\[
P(x, y, z) = \left[ Q(xz, y) \right]^N \left\{ \frac{1}{z} P \left( \frac{C(xz)}{Q(xz, y)}, \frac{C(y)}{Q(xz, y)}, z \right) + \left( 1 - \frac{1}{z} \right) p_0 U \left( \frac{C(y)}{Q(xz, y)} \right) \right\},
\]
where \( p_0 \) is the steady-state probability of an empty buffer. The \( U \)-function is defined as
\[
U(x) \triangleq \lim_{k \to \infty} \sum_{i=0}^{N} \Pr[t_{i-1} = j | s_k = 0] x^i.
\]
Unfortunately, we are not able to derive from (6) an explicit expression for \( P(x, y, z) \) or not even for \( S(z) = P(1, 1, z) \), which is the steady-state pgf of the buffer contents. In order to get more information about the behavior of the tagged output queue, we now consider only those values of \( x, y \) and \( z \) for which the arguments of the \( P \)-function on both sides of (6) are equal to each other, i.e., \( x = C(xz)/Q(xz, y) \) and \( y = C(y)/Q(xz, y) \). Using (3) and (4) it is easy to see that this is equivalent to
\[
x = \frac{(1 - \alpha) y}{1 - \alpha + \alpha y (1 - z)},
\]
and
\[
y = \frac{1 - \alpha + \alpha y}{\beta + \frac{1 - \beta}{N} [(N - 1) y + xz]}.
\]
It turns out that this allows us to get enough information from the functional equation to derive upper bounds for the mean and the tail distribution of the system contents (see Sections 5 and 6). From (8) and (9), \( x \) and \( y \) can be solved in terms of \( z \). It is easily verified that for a given value of \( z \), \( y \) satisfies a third-order equation. Hence, there are three sets of solutions. Here we only select a set of solutions which has the additional property that \( x = y = 1 \) for \( z = 1 \), which is denoted by \( \chi(z) \) and \( \xi(z) \). Choosing \( x = \chi(z) \) and \( y = \xi(z) \) in (6) yields a linear equation for the function \( P(\chi(z), \xi(z), z) \), which has the following solution:
\[
P(\chi(z), \xi(z), z) = \frac{(z - 1) p_0 \varphi_1(z) G(z)}{z - G(z)},
\]
where
\[
G(z) \triangleq [Q(\chi(z)z, \xi(z))]^N = \left( \beta + \frac{1 - \beta}{N} [(N - 1) \xi(z) + \chi(z)z] \right)^N
\]
and
\[
\varphi_1(z) \triangleq U(\xi(z)).
\]
From these definitions, we have \( G(1) = 1 \) and \( \varphi_1(1) = 1 \). The unknown parameter \( p_0 \) can be determined from the normalization equation \( P(\chi(z), \xi(z), z)|_{z=1} = 1 \). By using de l'Hospital's rule, we find \( p_0 = 1 - G'(1) \). However, in Eq. (10), \( \varphi_1(z) \) is unknown. Nevertheless, it is possible to derive upper
bounds for the mean and the tail distribution of the buffer occupancy, as we will describe in the following.

4. Unconditional and conditional arrival process in the steady state

Let \( s \) denote the number of cells stored in the tagged output queue just after a slot in the steady state, while \( e \) and \( v \) represent the number of arriving cells in the tagged output queue or non-tagged output queues respectively, during that slot. The two-dimensional joint pgf of the steady-state random variables \( e \) and \( v \) is defined as \( N(x, y) = E[x^e y^v] \). As \( e \) and \( v \) can be considered as the (steady-state) numbers of inlets (of the switching element) in state A or B2 respectively, \( N(x, y) \) can be expressed as

\[
N(x, y) = (p(B1) + p(A)x + p(B2)y)^N,
\]

where \( p(A) \), \( p(B1) \) and \( p(B2) \) denote the steady-state probabilities of finding an inlet in state A, B1 or B2 respectively. These probabilities can easily be found by solving the balance equations for the Markov chain in Fig. 1. As a result we obtain

\[
N(x, y) = \left(1 - p + \frac{p}{N} x + \frac{(N-1)p}{N} y\right)^N,
\]

which is also intuitively clear. Equation (13) describes the arrival process in the steady state. The marginal pgf \( E(x) \) of \( e \) can be obtained by setting \( y = 1 \) in (13). The marginal pgf \( V(y) \) of \( v \) can be derived by setting \( x = 1 \) in (13).

Next, we consider the unknown conditional probability mass function \( Pr[v = j | s = 0] \), appearing in (7). Since \( s = 0 \) also implies that \( e = 0 \), it is reasonable to think that the difference between the arrival processes to the non-tagged output queues observed when \( s = 0 \) or when \( e = 0 \) respectively, is small, i.e.,

\[
Pr[v = j | s = 0] \approx Pr[v = j | e = 0].
\]

Hence we might in a first instance replace the conditional probabilities \( Pr[v = j | s = 0] \) in (7) by the conditional probabilities \( Pr[v = j | e = 0] \). Numerical results later in the paper (see Figs. 2-4) show that this leads to a very accurate approximation. Furthermore, as \( s = 0 \) implies that few cells were sent to the tagged output queue during several previous slots, due to the correlated routing, we have

\[
Pr[v = j | s = 0] > Pr[v = j | e = 0] \quad \text{for large } j.
\]

Using Eq. (14) in (7), with Eqs. (12) and (13), then yields

\[
\varphi_s(z) \approx \varphi_v(z) \triangleq \sum_{j=0}^{N} Pr[v = j | e = 0][\xi(z)]^j = \frac{N(0, \xi(z))}{(1 - p + \frac{(N-1)p}{N} \xi(z))^N} = \left(1 - p + \frac{(N-1)p}{N} \xi(z)\right)^N.
\]

On the other hand, it is clear that, under the condition that \( s = 0 \), for large \( j \), the probability that \( j \) cells are sent to the non-tagged output queue is larger than in the unconditional case. That is,

\[
Pr[v = j | s = 0] > Pr[v = j] \quad \text{for large } j.
\]

Substituting \( Pr[v = j | s = 0] \) in (7) by \( Pr[v = j] \), we obtain from (12)

\[
\varphi_s(z) \approx \varphi(z) \triangleq \sum_{j=0}^{N} Pr[v = j][\xi(z)]^j = N(1, \xi(z)) = \left(1 - \frac{(N-1)p}{N} + \frac{(N-1)p}{N} \xi(z)\right)^N.
\]

(18)
5. Upper bounds of the mean buffer occupancy in the steady state

As mentioned before, the pgf $S(z)$ of $s$ can be expressed as $S(z) = P(1, 1, z)$. In order to obtain the mean buffer occupancy $E[s]$, we evaluate the total derivative of (10) with respect to $z$ at $z = 1$. This leads to

$$
\frac{dP}{dz} \bigg|_{z=1} = \varphi'(1) + G'(1) + \frac{G''(1)}{2[1 - G'(1)]},
$$

(19)

where

$$
\frac{dP}{dz} = \frac{\partial P}{\partial \chi} \frac{d\chi}{dz} + \frac{\partial P}{\partial \xi} \frac{d\xi}{dz} + \frac{\partial P}{\partial z}.
$$

Here $d$’s are used to indicate total derivatives and $\partial$’s to indicate partial derivatives. Since $\chi'(1) = \xi'(1) = 1$, we have

$$
\frac{\partial P}{\partial \xi} \bigg|_{z=1} = \chi'(1) = E'[s], \quad \frac{\partial P}{\partial \chi} \bigg|_{z=1} = E'(1) \quad \text{and} \quad \frac{\partial P}{\partial z} \bigg|_{z=1} = V'(1).
$$

So Eq. (19) yields

$$
E[s] = \varphi'(1) + G'(1) + \frac{G''(1)}{2[1 - G'(1)]} - E'(1)\chi'(1) - V'(1)\xi'(1).
$$

(20)

Here $G'(1)$ and $G''(1)$ can be expressed in terms of $\chi'(1), \xi'(1), \chi''(1)$ and $\xi''(1)$ by using (11), whereas these derivatives of $\chi(z)$ and $\xi(z)$ can be derived from Eqs. (8) and (9). We note in particular that $G'(1) = p$, so that $p_0 = 1 - p$, which is expected. Hence, the only unknown term left in (20) is $\varphi'(1)$, where $\varphi(z)$ is defined in (12). Although we are not able to obtain $\varphi(z)$, two upper bounds of $E[s]$ can be derived as follows.

From the definition of the $U$-function in (7), $(dU/d\xi) \bigg|_{z=1} - \sum_{j=0}^{N} \Pr[v = j | s = 0]$ is the average number of cell arrivals during a slot to the non-tagged output queues when $s = 0$. The inequality in (15) shows that $(dU/d\xi) \bigg|_{z=1}$ is larger than $\sum_{j=0}^{N} \Pr[v = j | \rho = 0]$. Since $\varphi'(1) = I(\xi'(1))$ and $\xi'(1) = -p/(1 - \alpha)N < 0$, we have $\varphi'(1) < \varphi'(1)$. Using this inequality in (20) leads to the following upper bound for the mean buffer occupancy:

$$
E[s] < S_1 \triangleq p + \frac{(N - 1)p^2}{2N(1 - \rho)} \left( 1 + 2 \frac{K + p - 1}{1 - \rho} + \frac{2pN}{N - \rho} \right).
$$

(21)

Next, from (17), $(dU/d\xi) \bigg|_{z=1}$ is also larger than $U'(1)$, the mean value of $\nu$. Since $\xi'(1) < 0$, this implies that $\varphi'(1) < \varphi'(1)$. Based on this inequality, a second upper bound of $E[s]$ can be obtained, i.e.,

$$
E[s] < S_2 \triangleq p + \frac{(N - 1)p^2}{2N(1 - \rho)} \left( 1 + 2 \frac{K + p - 1}{1 - \rho} \right).
$$

(22)

Comparing (21) with (22), we see that $S_2$ is always larger than $S_1$.

6. Upper-bound tail distribution of the buffer occupancy

It has been observed that, for a wide range of discrete-time queueing systems, the tail distribution of the buffer occupancy has a geometric form, i.e., for some threshold $T$,

$$
\Pr[s = n] \equiv By^n, \quad n > T.
$$

(23)
Numerical results have revealed that this is also true for the queueing system under study, as can be seen for instance from Fig. 4. In this section we will present an analytical approach to obtain the geometric decay rate $\gamma$ and upper bounds for the coefficient $B$. In this way, we can derive an upper-bound tail distribution of the queue length.

6.1. The geometric decay rate $\gamma$

From the inversion formula for $z$-transforms it follows that $\Pr[s = n]$ can be expressed as a weighted sum of negative powers of the poles of $S(z)$. Since the modulus of all these poles is larger than one, it is obvious that for large $n$, $\Pr[s = n]$ is dominated by the contribution of the pole with the smallest modulus. Let us denote this dominating pole by $z_0$. To ensure that the tail distribution is nonnegative anywhere, $z_0$ must necessarily be real and positive. Furthermore, we assume that $z_0$ has multiplicity one. The obtained results prove that this assumption is correct. Therefore, with respect to the asymptotic behavior of the buffer occupancy, $S(z)$ can be approximated as

$$S(z) \equiv \frac{\theta}{z - z_0}, \quad (24)$$

where $\theta$ is the residue of $S(z)$ in the point $z = z_0$. Taking the inverse $z$-transform of (24) thus gives rise to the following asymptotic result:

$$\Pr[s = n] \approx -\frac{\theta}{z_0} \left( \frac{1}{z_0} \right)^n, \quad n > T. \quad (25)$$

Comparing (25) to (23), we have

$$\gamma = \frac{1}{z_0} \quad \text{and} \quad B = -\theta. \quad (26,27)$$

As in [21,24,28], it can be argued that $z_0$ is also the pole with the smallest modulus of $P(x(z), \xi(z), z)$. Hence, in view of (10) and (11), $z_0$ is a real root of

$$\left( \beta - \frac{1 - \beta}{N} \right. \left[ (N - 1) \xi(z) + x(z) z \right] \right)^N = 0. \quad (28)$$

This can even be rigorously proved. As all sources are statistically independent, $G(z)$ given in (11) is the Perron–Frobenius eigenvalue of the aggregated arrival process to the tagged output queue [30]. So the dominant pole $z_0$ of $S(z)$ is determined by $z - G(z) = 0$ [29]. Furthermore this is also confirmed by numerical results (see for instance Fig. 4). The pole $z_0$ can be obtained numerically by using, for instance, the Newton–Raphson algorithm, where in each step $\chi(z)$ and $\xi(z)$ are calculated from (8) and (9). Selecting the correct solution of (8) and (9) is one of the problems here. It can be solved on the basis of the observation (not proved here) that $0 < \xi(z) < 1$ for all real $z > 1$. Finally, we note from Eqs. (8), (9) and (28) that the unknown conditional probability mass function $\Pr[\nu = j | s = 0]$ has no influence on $z_0$, which means that $\gamma$ can be calculated exactly.

6.2. Upper bounds for the coefficient $B$ of the geometric form

When the number of cells stored in the tagged output queue just after a given slot is sufficiently large ($\gg N$), we may think that the number of cell arrivals during this slot (which is not larger than $N$) has nearly no impact on the total buffer occupancy. That is, if $n$ is sufficiently large ($n > T$), we may assume
that the conditional probabilities \( \Pr[e = i, v = j | s = n] \) are nearly independent of \( n \), and approach to some limiting values for \( n \to \infty \), denoted by \( \omega(i, j) \), with corresponding joint pgf \( \Omega(x, y) \), i.e.,

\[
\Pr[e = i, v = j | s = n] \approx \omega(i, j), \quad n > T.
\]  

(29)

Now, let \( \pi(i, j | k, l) \) denote the one-step transition probability that there are \( i \) cell arrivals in the tagged output queue and \( j \) cell arrivals in the non-tagged output queues in the current slot, given that there were \( k \) and \( l \) cell arrivals respectively in the previous slot. Then, we have (for large \( n \))

\[
\Pr[e = i, v = j | s = n] = \frac{\Pr[e = i, v = j, s = n]}{\Pr[s = n]}
\]

\[
= \sum_{k=0}^{N} \sum_{l=0}^{N-k} \pi(i, j | k, l) \Pr[e = k, v = l, s = n + 1 - i]
\]

\[
\Pr[s = n]
\]

Taking the limit for \( n \to \infty \), and using Eqs. (23), (26) and (29), the above equation leads to

\[
z_0 \omega(i, j) = z_0^i \sum_{k=0}^{N} \sum_{l=0}^{N-k} \pi(i, j | k, l) \omega(k, l), \quad 0 \leq i \leq N, \quad 0 \leq j \leq N - i.
\]  

(30)

From (30), the following equation for the pgf \( \Omega(x, y) \) can then be derived:

\[
z_0 \Omega(x, y) = \left[ Q(xz_0, y) \right] \Omega \left( \frac{C(xz_0)}{Q(xz_0, y)}, \frac{C(y)}{Q(xz_0, y)} \right).
\]  

(31)

As can be expected intuitively, it is possible to show that the solution \( \Omega(x, y) \) of (31) has the same form of expression as \( N(x, y) \). Specifically, \( \Omega(x, y) \) can be expressed as

\[
\Omega(x, y) = (1 - \sigma_1 - \sigma_2 + \sigma_1 x + \sigma_2 y)^N,
\]  

(32)

where \( \sigma_1 \) and \( \sigma_2 \) are the (conditional) probabilities of finding an inlet in state A or B2 respectively, when the number of cells in the tagged output queue is extremely large. Using Eqs. (31) and (32), \( \sigma_1 \) and \( \sigma_2 \) can be derived explicitly as

\[
\sigma_1 = \frac{(1 - \beta) z_0 (z_0^{1/N} - \alpha)}{N(z_0^{1/N} + 1 - \alpha - \beta)(z_0^{1/N} - \alpha z_0) + (1 - \beta)(z_0 - 1)z_0^{1/N}}
\]

and

\[
\sigma_2 = \frac{(1 - \beta)(N - 1)(z_0^{1/N} - \alpha z_0)}{N(z_0^{1/N} + 1 - \alpha - \beta)(z_0^{1/N} - \alpha z_0) + (1 - \beta)(z_0 - 1)z_0^{1/N}}.
\]

So the joint pgf \( \Omega(x, y) \) can be obtained analytically.

Using Eq. (29), the joint pgf \( P(x, y, z) \) can now be approximately expressed as

\[
P(x, y, z) \approx \sum_{j=0}^{N-i} \sum_{l=0}^{N-k} \Pr[e = i, v = j, s = n]x^iy^z + \Omega(x, y) \sum_{n=0}^{T} \Pr[s = n]z^n.
\]  

(33)

For \( x = \chi(z) \) and \( y = \xi(z) \), we know that \( z_0 \) is a pole of both the \( P \)-function and \( S(z) \). Since \( T \) is finite, multiplying by \( (z - z_0) \) on both sides of Eq. (33) and taking the \( z \to z_0 \) limit, yields

\[
\theta = \frac{\eta}{\Omega(\chi(z_0), \xi(z_0))}.
\]  

(34)
where $\theta$ is the residue of $S(z)$ at $z = z_0$, as defined in (24). The quantity $\eta$ can be obtained from Eq. (10) with de l'Hospital's rule as

$$\eta = \lim_{z \to z_0} (z - z_0) P(\chi(z), \xi(z), z) = \frac{(z_0 - 1)(1 - p) \varphi(z_0) G(z_0)}{1 - G'(z_0)}. \tag{35}$$

Finally, from (27), (34) and (35), we find that the coefficient $B$ is given by

$$B = \frac{(z_0 - 1)(1 - p) \varphi(z_0)}{[G'(z_0) - 1] \Omega(\chi(z_0), \xi(z_0))}. \tag{36}$$

In Eq. (36), the quantity $\varphi(z_0)$ is unknown. However, as mentioned before, it can be shown that $0 < \xi(z) < 1$ for all real $z > 1$. In particular, this is also true for $z = z_0$. Therefore, from the inequalities in (15) and (17), and the definitions of $\varphi(z)$ and $\varphi(z)$ in (16) and (18), it follows that $\varphi(z_0)$ is upper bounded by both $\varphi(z_0)$ and $\varphi(z_0)$. Using these two values as approximations for $\varphi(z_0)$ in (36) therefore yields two upper bounds for the coefficient $B$, denoted as $B_1$ and $B_2$ (corresponding to $\varphi(z_0)$ and $\varphi(z_0)$ respectively).

7. The cell delay

We define the delay of a cell as the number of slots between the end of its arrival slot and the end of the slot during which the cell is transmitted and thus leaves the output queue. In [26] and [27], the following relationship was established between the pgf of the system contents $S(z)$ and the pgf of the cell delay $D(z)$, for discrete-time single-server queueing systems with general, possibly correlated, arrivals:

$$D(z) = \frac{S(z) - S(0)}{1 - S(0)} = \frac{S(z) - p_0}{1 - p_0}. \tag{37}$$

Using this equation, the moments of the cell delay can be derived in terms of the moments of the system contents. For instance, for the mean cell delay $E[d]$, we have

$$E[d] = D'(1) = \frac{S'(1)}{1 - p_0} = \frac{E[s]}{p}, \tag{38}$$

in agreement with Little's theorem. Using the upper bounds obtained for $E[s]$, we then get corresponding upper bounds for $E[d]$. Also, with (37), the tail distribution of the cell delay can be found from the tail distribution of the system contents. We have

$$\Pr[d = n] = \frac{\theta}{z_0} \left( \frac{1}{z_0} \right)^n \frac{1}{p} = \frac{B}{p} \gamma^n, \quad n > T. \tag{39}$$

That is, the tail distribution of the delay also has a geometric form, with the same decay rate as the tail distribution of the system contents.

8. Numerical versus analytical results

In order to check the accuracy of the analytical results derived in the previous sections, we have also analyzed the tagged output queue under the assumption of a finite, but "large" waiting room, using the same type of three-dimensional state description as in the analytical approach. In this case, however, a
numerical solution of the resulting balance equations, rather than a solution in terms of generating functions, was performed. As the dimension of the set of balance equations grows rapidly with the number of inlets/outlets $N$ of the switch and the (finite) size of the tagged output queue, the comparison between numerical and analytical results was only practicable for small switch sizes and limited buffer sizes, combined with low to intermediate traffic loads and relatively small values for the burstiness factor, in order that the finite buffer size can be considered "large". Some results are presented below for $N = 4$ and various values of $p$ and $K$.

In Fig. 2, we have plotted the analytical upper bounds $S_1$ and $S_2$ for the mean buffer contents, as well as numerical results, versus the load $p$, for $K = 1$ and $K = 2$. As one can see, $S_1$ is very close to the numerical results, which could be expected, based on Eq. (14). It is also clear that the difference between $S_2$ and the numerical results becomes larger as $p$ and $K$ increase, whereas $S_1$ is also accurate for large $p$ and $K$. Similar conclusions can be drawn for the mean cell delay $E[d]$, as is illustrated in Fig. 3.

In Fig. 4, we compare the upper-bound tail distribution of the queue length given by $B_1 \gamma^n$ with numerical results, for $K = 2$ and various values of the load $p$. The figure illustrates that the approximate method described above gives a very tight upper bound for the tail of the distribution. Moreover, the numerical results confirm that the analytical approach to obtain $z_0$ and $\gamma$ is correct. In Table 1, we compare the derived upper bounds for the coefficient $B_1$, with numerical results. As expected, $B_1$ is smaller than $B_2$. Furthermore, $B_1$ is very close to the numerical results.

In our analytical approach, discussed in the previous sections, we have assumed an unlimited storage capacity for the (tagged) output queue. In practice, however, buffers are always of finite size, and a fraction of the arriving cells will be lost. It is important to be able to predict this so-called cell loss ratio (CLR) for a switch with a given configuration and traffic parameters, and a prescribed buffer size $S$ for the output queues. One way to deal with this problem is to solve the balance equations for the system by numerical means, and, from this, calculate the exact CLR-values. However, as indicated before, this becomes extremely memory and/or time consuming (and error prone) for high values of $N$ and $S$. In
order to overcome this difficulty, we have devised a heuristic approach to predict the cell loss ratio for a finite buffer from the tail distribution of the buffer contents in an unlimited capacity queue. Specifically, our heuristic formula is based on the observation that the overflow probability in a continuous-time

![Graph showing mean cell delay versus load p: upper bounds and numerical results, for N = 4 and K = 1, 2.](image1)

![Graph showing Pr[s = n] versus n: upper bound 1 and numerical results, for N = 4, K = 2 and p = 0.2, 0.4, 0.6.](image2)
Table 1
Coefficient $B$ of the geometric form: upper bounds and numerical results, for $N = 4$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$K$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1</td>
<td>0.36950</td>
<td>0.37472</td>
<td>0.36884</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>0.24030</td>
<td>0.24991</td>
<td>0.23841</td>
</tr>
<tr>
<td>0.6</td>
<td>1</td>
<td>0.13393</td>
<td>0.14162</td>
<td>0.13246</td>
</tr>
<tr>
<td>0.4</td>
<td>2</td>
<td>0.05076</td>
<td>0.05232</td>
<td>0.05052</td>
</tr>
<tr>
<td>0.6</td>
<td>2</td>
<td>0.04853</td>
<td>0.05093</td>
<td>0.04814</td>
</tr>
</tbody>
</table>

Table 2
Cell loss ratio and $\Pr[s > S](1 - p)/p$ for $N = 4$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$K$</th>
<th>$S$</th>
<th>CLR</th>
<th>$\Pr[s &gt; S](1 - p)/p$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Upper bound 1</td>
<td>Upper bound 2</td>
</tr>
<tr>
<td>0.2</td>
<td>1</td>
<td>8</td>
<td>1.216008E-7</td>
<td>1.587516E-7</td>
</tr>
<tr>
<td>0.4</td>
<td>1</td>
<td>15</td>
<td>1.248172E-6</td>
<td>1.846438E-6</td>
</tr>
<tr>
<td>0.6</td>
<td>1</td>
<td>30</td>
<td>2.335455E-5</td>
<td>3.416390E-5</td>
</tr>
<tr>
<td>0.8</td>
<td>1</td>
<td>20</td>
<td>4.259122E-2</td>
<td>4.614403E-2</td>
</tr>
<tr>
<td>0.2</td>
<td>2</td>
<td>15</td>
<td>8.780028E-6</td>
<td>1.009771E-5</td>
</tr>
<tr>
<td>0.4</td>
<td>2</td>
<td>40</td>
<td>4.700978E-7</td>
<td>8.483441E-7</td>
</tr>
<tr>
<td>0.6</td>
<td>2</td>
<td>30</td>
<td>3.050019E-3</td>
<td>4.771488E-3</td>
</tr>
<tr>
<td>0.8</td>
<td>2</td>
<td>30</td>
<td>6.405418E-2</td>
<td>6.552858E-2</td>
</tr>
</tbody>
</table>

The M/M/1/S queueing system, for high values of $S$, is nearly equal to the product of the probability of having more than $S$ customers in an M/M/1 system, and a "correction factor" equal to $(1 - p)/p$, where $p$ is the load. Using the same formula in the current context, we thus approximate the cell loss ratio for an output buffer of size $S$ as follows:

$$\text{CLR} \equiv \frac{1 - p}{p} \cdot \frac{\Pr[s > S]}{1 - \gamma} \cdot \gamma^{S+1}.$$  \hspace{1cm} (40)

where, in view of (23),

$$\Pr[s > S] = \frac{B}{1 - \gamma} \gamma^{S+1}.$$  \hspace{1cm} (41)

In Table 2, we compare the actual cell loss ratio (obtained by numerically solving a set of balance equations) with the heuristic in (40)–(41), for various values of $p$, $K$ and $S$, using either $B_1$ or $B_2$ as an upper bound for $B$ in Eq. (41). The results in Table 2 show that our heuristic approach leads to estimates of the CLR which, in general, are somewhat higher than the actual CLR, and, for realistic values of the load ($p = 0.8$), are even quite close. We therefore believe that this approach can be very useful in practice for buffer dimensioning purposes.

9. Discussion

Having demonstrated in the previous section the validity of the analytic techniques developed in this paper, we now present some further results, obtained by applying these techniques in the range of parameters where a numerical approach is impractical. Fig. 5 shows the mean buffer occupancy in an
output queue versus the load, for a switch with 16 inlets and outlets, for various values of the burstiness factor $K$ of the sources. From the position of the curves in this figure, it is clear that, for a given value of the mean load $p$, the burstiness of the sources has a tremendous impact on the mean number of cells in

Fig. 5. Upper bound 1 of the mean buffer contents versus load $p$, for $N = 16$ and $K = 1, 2, 5, 10, 20, 50, 100$.

Fig. 6. Upper bound 1 of $\Pr[s > n]$ versus $n$, for $N = 16$, $p = 0.6$ and $K = 1, 2, 5, 10, 100$. 

Fig. 7. Mean buffer contents versus load $p$ for correlated routing (upper bound 1) and uncorrelated routing (exact), for $N = 16$ and $K = 1, 10$.

Fig. 8. $\Pr[s > n]$ versus $n$ for correlated routing (upper bound 1) and uncorrelated routing (exact), for $N = 16$, $p = 0.8$ and $K = 1, 10$. 

the output queue. Specifically, we note that the congestion in an output queue may be seriously underestimated if a Bernoulli arrival process is used as an approximation for bursty traffic, since Bernoulli arrivals correspond to $K = 1$, while bursty sources will typically give rise to much higher values of $K$. Similar conclusions can be drawn from Fig. 6, where we have plotted the cumulative tail probabilities $\Pr[s > n]$ of the output buffer occupancy versus $n$, at a given load $p = 0.6$, for different burstiness factors $K$.

In Figs. 7 and 8, we compare the correlated routing mechanism investigated in this paper, to the case of uncorrelated routing, which was studied in [22], for a switching element with 16 inlets and outlets. Specifically, Fig. 7 shows the mean output-queue contents versus the load, and Fig. 8 shows the tail probabilities $\Pr[s > n]$ versus $n$, at a load $p = 0.8$. Several conclusions can be drawn from Figs. 7 and 8. First, it can be observed that the required buffer space in an output queue is always much higher in the case of correlated routing than in the case of uncorrelated routing, regardless of the burstiness factor $K$ of the sources, although the performance deteriorates more if the sources are more bursty. Second, the influence of the burstiness factor $K$ on the level of congestion in an output queue is much more pronounced in case of correlated routing than for uncorrelated routing. This can be intuitively understood by the observation that independent destination addresses from cell to cell more or less “destroy” most of the burstiness of the arrival stream between the inlets of the switching element and the entrance of an output queue, while a correlated routing mechanism in some sense simply passes the burstiness of the sources to the entrance of the output queues. Third, routing correlation has a more substantial impact on the queueing behavior of a switch than input correlation (i.e., burstiness of the sources), although, in general, we may conclude that both types of correlation amplify each other’s effect.

10. Conclusions

In analyzing the performance of an ATM switching element with bursty sources and correlated routing, we have derived explicit upper bounds for the means and tail distributions of the system contents of a tagged output queue and the delay of a cell. These upper bounds were obtained under the assumption of an infinite-capacity queue, by combining a generating-functions approach with approximation techniques. Comparison with numerical results shows that the obtained upper bounds are very tight, and can even be used to predict cell loss ratios in finite-capacity buffers. We have also observed that the queueing performance of the switching element deteriorates as the burstiness of the sources and/or the amount of correlation in the routing process get higher.

Acknowledgements

The first author wishes to thank the Belgian National Fund for Scientific Research (N.F.W.O.) for support of this research.

References

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