Exact Message Delay Derivation for TDMA Schemes with Multiple Contiguous Outputs and General Independent Arrivals

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1. Introduction

TDMA (Time Division Multiple Access) models have been extensively investigated during the past several years [1]–[5]. They are used to describe situations in which several stations use the same communication line to send their messages, by using dedicated time slots.

The queue-length analysis (at a given station) in a TDMA scheme, where several contiguous slots are allocated to the stations, has been thoroughly treated by Rubin and Zhang [1] for the case of a general uncorrelated arrival process. An even more general queuing model, of which the present model is a special case, has been similarly analysed by Bruneel [6]. Rubin [3] has found an expression of the probability generating function of the packet delay, for the case where one single slot is allocated per frame to each station. For the case of multiple slot allocations per frame, Ko and Davis [4] found an approximate expression for the mean packet delay assuming a Poisson arrival process. Using a similar approximation, Bruneel [2] obtained an expression for the whole probability generating function of the packet delay, valid for a general uncorrelated arrival process. Lee and Liang [5] extended Bruneel’s technique to a slightly different model. Rubin and Zhang [1], finally, have obtained an exact solution for the generating function of the packet delay, for a Bernoulli and a geometric arrival process, in which cases one can use the memoryless properties of these distributions to simplify the analysis considerably.

In this paper, we derive an expression for the probability generating function of the packet delay in a TDMA scheme with a generally distributed arrival stream. To our best knowledge, no work that presents such an analysis has been published yet. The outline of the paper is as follows. In Section 2, we introduce the TDMA model in use. In Section 3, the queue length is studied, and some results that will be used in the following analysis of the packet delay are presented. Next, we obtain the probability generating function of the packet delay, for a general arrival process in Section 4. In Section 5, we discuss the expected value of the packet delay, whereas in Section 6, we confine ourselves to a memoryless arrival stream, in which case we can compare our results with the ones found by Rubin and Zhang. Finally, some numerical results are given in Section 7.

2. The model

In a TDMA scheme, several stations wish to send messages over the same channel. Every message is divided into an integer number of packets of equal length, while the time axis is divided into slots of fixed length, which is chosen in such a way that exactly one
packet can be sent during one slot. Due to the synchronous transmission mode in a TDMA scheme, the transmission of a packet always starts at the beginning of a slot.

The number of packet arrivals into the buffer at a given station during the consecutive slots is described by a sequence of random variables that are considered to be i.i.d. This means that we can define a generic random variable \( e \) (with probability generating function \( E(z) \)) to describe the number of packet arrivals during an arbitrary slot.

A number of contiguous slots are allocated to each station, during which it can send its messages. When we concentrate on one station, there is a period consisting of a constant number of slots (called \( L \)) during which the station cannot send messages over the communication line. We will refer to this period as the B-period (Blocked) of the station, and its \( L \) slots are called B-slots. After a B-period follows a constant number of slots (called \( N \)) during which the channel is available to the station. We will refer to this period as the A-period (Available) of the station, and its \( N \) slots are called A-slots. Finally, we define a frame as the time interval between the starts of two adjacent B-periods. As shown in Fig. 1, all the frames consist of a B-period followed by an A-period, and they are all identical.

![Diagram of time axis into slots, A-periods, B-periods and frames.](image)

Fig. 1. Division of the time axis into slots, A-periods, B-periods and frames.

In the following, we assume that the system under study reaches a steady state after a sufficient period of time. The condition that must be satisfied to achieve this, is that the average number of packets entering the station during a slot is strictly less than the maximum number of packets that can leave the station during a randomly chosen slot:

\[
E'(1) < \frac{N}{N+L},
\]

(1)

(where the prime denotes the first derivative with respect to the argument). If this condition is met, the system will reach a stochastic equilibrium.

3. The queue length

Most of the results in this section are based upon Rubin and Zhang [1]. We define \( v_k \) (with corresponding probability generating function \( V(z) \)) as the queue length in the station at the beginning of the \( k \)-th slot of a frame in the steady state of the system, assuming that such a state has been reached. We can write down the following relationships between the \( v_k \)'s:

\[
v_k = v_{k-1} + e_{k-1}, \quad 2 \leq k \leq L + 1
\]

\[
v_k = (v_{k-1})^* + e_{k-1}, \quad L + 2 \leq k \leq L + N
\]

\[
v_k = (v_{L+N-1})^* + e_k, \quad 2 \leq k \leq L + N
\]

where \((...)^*\) denotes max\((0,...)\), and \(e_k\) indicates the number of packet arrivals during he \( k \)-th slot of a frame.

The meaning of these relations is clear: the number of packets waiting in the station to be sent, at the beginning of the \( k \)-th slot of a frame, is the sum of the queue length at the beginning of the previous slot and the number of packets that have entered the system during the \((k - 1)\)-th slot, with the restriction that one packet will leave the station if the \((k - 1)\)-th slot was an A-slot and the station was not empty at the beginning of this slot. As \( v_k \) and \( e_k \) (with probability generating functions \( V_k(z) \) and \( E(z) \)) respectively are statistically independent for all \( k \), we find the following relationships between the corresponding generating functions:

\[
V_k(z) = E[z^k] = E(z) V_{k-1}(z), \quad 2 \leq k \leq L + 1
\]

\[
V_k(z) = z^{-1} E(z) (V_{k-1}(z) + (z-1) V_{k-1}(0)), \quad L + 2 \leq k \leq L + N
\]

\[
V_k(z) = z^{-1} E(z) (V_{L+N-1}(z) + (z-1) V_{L+N}(0)),
\]

where \(E[.]\) denotes the expected value of the argument. Using these relations iteratively, we find the following expressions for \( V_L(z) \) and \( V_{L+N}(z) \), the probability generating functions of the queue length at the beginning of a B-period and an A-period respectively:

\[
V_L(z) = \frac{(z-1) E(z)^N}{z^N - E(z)^N + L \sum_{k=1}^{N} (z E(z) - 1)^{k-1} V_{L+k}(0)},
\]

\[
V_{L+N}(z) = E(z)^N V_L(z).
\]

Furthermore, the normalization condition of these generating functions implies:

\[
\sum_{k=1}^{N} V_{L+k}(0) = N - (N + L) E'(1).
\]

(5)

In eq. (4), \( V_L(z) \) and \( V_{L+N}(z) \) are given in terms of the \( N \) unknown constants \( V_{L+k}(0) \). These constants can be eliminated from (4) by using Rouché’s theorem and the fact that the generating functions \( V_L(z) \) and \( V_{L+N}(z) \) are analytic in the unit disk \( \{z: |z| \leq 1\} \) of the complex plane. Rubin and Zhang [1] have shown that one can write:

\[
\sum_{k=1}^{N} \frac{(z E(z) - 1)^{k-1} V_{L+k}(0)}{z^k - z E(z)^k} = (N - (N + L) E'(1) \prod_{j=1}^{N-1} \frac{1 - z_j}{1 - z E(z_j)}),
\]

(6)

with \( z_j \) the \( N-1 \) zeroes of \( z^N - E(z)^N + L \) inside the complex circle \( \{z: |z| < 1\} \).

We define \( v \) (with corresponding probability generating function \( V(z) \)) as the random variable that de-
scribes the queue length at the beginning of a randomly chosen slot. We can write:

\[ V(z) = \frac{1}{N + L} \sum_{k=1}^{N+L} V_k(z). \]

From the expressions (3) and (4), and the normalization condition (5), we find after some algebraic manipulations:

\[
V(z) = \frac{(z-1)^2 (1 - E(z)) E(z)^{N+1}}{N + L \left( z^{N-L} - E(z)^{N+L} (z - E(z)) (1 - E(z)) \right)} + \sum_{k=1}^{N} \left( \frac{z E(z)^{N-k-1}}{N + L} V_{k+1}(0) + \frac{N - (N + L) E'(1)}{z - E(z)} \right). \tag{7}
\]

Again, we can eliminate the unknown constants \( V_{k+1}(0) \) out of this expression, using eq. (6). The expected value of the queue length at the beginning of a randomly chosen slot can then be obtained as:

\[
E[z] = V'(1) = \frac{L}{2(N + L)} \sum_{j=1}^{N-1} E(z_j) + z_j + E'(1) + \frac{(N + L) E''(1) + LE'(1)}{2(N - (N + L) E'(1))}. \tag{8}
\]

### 4. The packet delay

In this section, we derive an expression for the generating function of the packet delay in the TDMA scheme under study. From now on, we consider the service discipline in the queue as being first-come-first-served.

![Fig. 2. The packet delay in a TDMA scheme.](image)

As shown in Fig. 2, the packet delay is defined as the number of slots between the end of the arrival slot of the packet, and the end of the slot when the packet leaves the station. We define the following random variables (with their corresponding probability generating functions mentioned between parentheses):

1. \( d_k(D_k(z)) \) describes the packet delay for a randomly chosen packet that arrives at the station during the \( k \)-th slot of a frame.
2. \( u_k(U_k(z)) \) describes the number of packets that must leave the station before a tagged randomly chosen packet is at the head of the queue and will be the next packet to be transmitted, observed at the moment of arrival of this packet at the station during the \( k \)-th slot of a frame.
3. \( f(F(z)) \) is the number of packets that have arrived in the station during the same slot as the tagged packet, and that will leave the station before this packet.

In Bruneel [7], it is shown that the following relationship exists between \( F(z) \) and \( E(z) \):

\[
F(z) = \frac{E(z) - 1}{(z-1) E'(1)}. \tag{9}
\]

The quantities \( v_k \), \( u_k \) and \( f \) are related as follows:

\[
u_k = f + v_k. \tag{10a}
\]

As \( v_k \) and \( f \) are statistically independent, this yields

\[
U_k(z) = F(z) V_k(z). \tag{10b}
\]

Now we derive an expression for the probability generating function of the packet delay for a packet arriving during the \( k \)-th slot of a frame. We must consider two cases:

#### Arrival during a B-slot \( (1 \leq k \leq L) \).

The \( u_k \) packets in front of our randomly chosen packet must leave the station before the latter can be transmitted. Observe that we can express the positive integer random variable \( u_k \) in a unique way in terms of two discrete random variables \( b_k \) and \( c_k \), as follows:

\[
u_k = c_k N + b_k \quad \text{with} \quad \begin{cases} 0 \leq c_k \\ 0 \leq b_k \leq N - 1 \end{cases}.
\]

The random variable \( c_k \) describes the number of complete frames in the packet delay, while the random variable \( b_k \) describes the number of packets that will leave the buffer before the tagged packet during the A-period in which the tagged packet leaves the buffer. As shown in Fig. 3, we thus find:

\[ d_k = L - k + (N + L) c_k + b_k + 1. \]

This implies that we can write

\[
D_k(z) = z^{L-k+1} \sum_{i=0}^{N-1} \sum_{j=0}^{i+j} z^{(N+L)i+j} \text{Prob} \left[ U_k = N i + j \right].
\]

![Fig. 3. Components of the packet delay for a packet arriving during a B-slot.](image)
We obtain
\[
D_k(z^N) = z^{N(L-k+1)} \cdot \\
\sum_{j=0}^{N-1} \sum_{i=0}^{\infty} z^{N+i+j} \cdot \text{Prob}[u_k = N i + j] = \\
z^{N(L-k+1)} \sum_{j=0}^{N-1} \sum_{n=0}^{\infty} z^{N+i+j} \cdot \text{Prob}[u_k = n] \cdot \\
\sum_{i=-\infty}^{\infty} \delta(n - N i - j).
\]
where \(\delta(.)\) denotes the Kronecker delta function. As in Bruneel [2], we use the identity
\[
\frac{1}{N} \sum_{i=0}^{N-1} a^{m i} = \sum_{k=-\infty}^{+\infty} \delta(m - N k)
\]
with \(a = \exp\left(\frac{2\pi i}{N}\right)\),
\[
(11)
\]
(with \(i\) the imaginary unit) to eliminate the Kronecker delta functions in the above expression of \(D_k(z^N)\):
\[
D_k(z^N) = z^{N(L-k+1)} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{\infty} (a^N z^{N+i+j}) \cdot \text{Prob}[u_k = n] = \\
\sum_{j=0}^{N-1} \sum_{n=0}^{\infty} (z^N a^N)^{-j}.
\]
Using relation (10b), we finally get
\[
D_k(z^N) = z^{N(L-k+1)} \frac{1}{N} \cdot \\
\sum_{n=0}^{N-1} \sum_{k=0}^{\infty} \frac{1 - z^{-NL}}{1 - a^N z^{-L}} F(a^N z^{N+i+j}) V_k(a^N z^{N+i+j}),
\]
where we have used the property \(a^N = 1\).

This expression determines the probability generating function of the packet delay for a packet arriving at the station during the \(k\)-th slot of a B-period, in terms of the arrival process and the queue-length distribution at the beginning of this slot.

arrival during an A-slot \((L + 1) \leq k \leq N + L\).

In this case, we must take into account that a packet will leave the station at the end of the slot of arrival of our tagged packet, only if the queue length at the beginning of this slot was nonzero. Therefore, we define the two discrete random variables \(c_a\) and \(b_a\) as
\[
u_k + (1 - v_k) + k - L - 1 = N c_a + b_a,
\]
with
\[
0 \leq c_a, 0 \leq b_a \leq N - 1,
\]
where \((...)^+\) denotes max \((0,...)\), as before. The quantity \(N c_a + b_a\) describes the number of A-slots between the beginning of the A-period during which the tagged packet enters the buffer, and the beginning of the slot during which this packet leaves the buffer. The random variable \(c_a\) describes the complete A-periods in this time interval, while \(b_a\) describes the remaining

A-slots. As shown in Fig. 4, the packet delay is given by
\[
d_k = (N + L) c_a + b_a + 1 - (k - L).
\]
We then obtain the following expression for the generating function of the packet delay:
\[
D_k(z) = z^{(L-k+1)} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j=0}^{\infty} z^{N(i+j)} \cdot \\
\cdot \text{Prob}[u_k = (N+1) i + n] = N i + j.
\]
This can be worked out in a complete analogous way as we did in the previous case. We find
\[
D_k(z^N) = z^{N(L-k+1)} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{\infty} \frac{1 - z^{-NL}}{1 - a^N z^{-L}} F(a^N z^{N+i+j}) V_k(a^N z^{N+i+j}) \cdot \\
\cdot E(\text{[}(a^N z^{N+i+j})(1 - v_k)^+\text{]}).
\]
Using (10a), we finally obtain
\[
D_k(z^N) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{\infty} \frac{1 - z^{-NL}}{1 - a^N z^{-L}} F(a^N z^{N+i+j}) V_k(a^N z^{N+i+j}) \cdot \\
\cdot [V_k(a^N z^{N+i+j}) + (a^N z^{N+i+j} + 1) V_k(0)].
\]
(13)

The property we are really interested in, is the packet delay in a TDMA scheme, independent of the position of the slot of arrival in a frame. If we denote by \(D\) (with corresponding generating function \(D(z)\)) the random variable describing the packet delay for a randomly chosen packet, we can write:
\[
D(z) = \frac{1}{N+L} \sum_{k=1}^{N+L} D_k(z).
\]
Using expressions (12) and (13) for \(D_k(z)\), we obtain
\[
D(z^N) = z^{N(L+1)-k} F(a^N z^{N+i+j}) \frac{1}{N(N+L)} \cdot \\
\sum_{k=1}^{N+L} \sum_{i=0}^{\infty} z^{N(L+1)-k} V_k(a^N z^{N+i+j}) \cdot \\
+ \sum_{k=1}^{N+L} \sum_{i=0}^{\infty} (a^N z^{N+i+j} - 1) V_k(0).
\]
With the use of the relations (3) between the various generating functions of the queue length at the begin-
ning of a slot, the expression (9) of \( F(z) \), and keeping in mind that \( a^n = 1 \), we obtain after some lengthy calculations

\[
D(z^N) = \frac{1}{N(N + L)} E'(1) \left( \sum_{s=0}^{N-1} \frac{1 - z^{-sL}}{1 - a^{-s} z^{-L}} E(a^s z^{N+s}) - 1 \right) \left( \sum_{k=1}^{N} (a^k z^k)^{N-1} V_{L+k}(0) \right). \tag{14}
\]

Again, the unknown constants \( V_{L+k}(0) \) can be eliminated out of the above expression. Indeed, from the developments leading to eq. (6), it can also be deduced that

\[
\sum_{k=1}^{N} (a^k z^k)^{N-1} V_{L+k}(0) = \left( N - (N + L) E'(1) \right) \frac{N-1}{1 - z L E(z)} \frac{z^N}{1 - z \frac{E(z)}{z}},
\]

where \( z \) (\( 0 < z < 1 \)) are the \( N-1 \) zeros of \( z^N - E(z) z^{N+L} \) inside the unit disk of the complex plane \( |z| < 1 \). We finally find

\[
D(z^N) = \frac{N - (N + L) E'(1)}{N(N + L) E'(1)} \left( \sum_{s=0}^{N-1} \frac{1 - z^{-sL}}{1 - a^{-s} z^{-L}} E(a^s z^{N+s}) - 1 \right) \left( \sum_{k=1}^{N} (a^k z^k)^{N-1} \right) \frac{z^N}{1 - z \frac{E(z)}{z}}. \tag{15}
\]

This is the generating function of the packet delay for the TDMA scheme under study. The shape of the arrival distribution is arbitrary, and the above expression holds for single arrivals of packets, as well as for group arrivals (bulks).

5. The expected value of the packet delay

We obtain an expression for the expected value of the packet delay, by taking the first derivative of the generating function given in (14), in the point \( z = 1 \) (again, primes are used to denote derivatives):

\[
D'(1) = -\frac{L(N-1)}{2N} + \frac{(N+L)E'(1) - 2(N+L)E'(1)^2 + (2N+L)E'(1)}{2(N-(N+L)E'(1))E'(1)} + \frac{L}{N(N-(N+L)E'(1))} \sum_{k=1}^{N} V_{L+k}(0) - \frac{L}{N(N+L)E'(1)} \sum_{s=1}^{N} \frac{a^s}{a^s - 1} \sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j}.
\]

The complex sum in the last term of this expression can be written as

\[
\sum_{s=1}^{N-1} \frac{a^s}{a^s - 1} = \left( \sum_{s=0}^{N-1} \frac{1}{a^s} \right) - \frac{1}{a^0 - 1}.
\]

The first term between brackets is evaluated using the identity (11), and for the last term we find

\[
\sum_{s=1}^{N-1} \frac{a^s}{a^s - 1} = \frac{N-1}{2} \left( \cos \frac{2\pi s}{N} - \frac{a^s}{a^s - 1} \right).
\]

We obtain

\[
\sum_{s=1}^{N-1} \frac{a^s}{a^s - 1} = \frac{N-1}{2} \left( \cos \frac{2\pi s}{N} - \frac{a^s}{a^s - 1} \right). \tag{16}
\]

On the other hand, by taking the first derivative of (6) in the point \( z = 1 \), we get

\[
\sum_{k=1}^{N} (k-1) V_{L+k}(0) = \frac{N(N+L) E'(1) \sum_{j=1}^{N-1} \frac{E(z_j)}{E(z_j) - z_j}}{2(N-(N+L)E'(1))}.
\]

Using these last two results in the above expression for the mean packet delay, we finally obtain:

\[
D'(1) = \frac{1}{E'(1)} \left( \frac{L}{2(N+L)} \sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} + \frac{N+L}{E'(1) + L E'(1)} \right). \tag{17}
\]

Comparing this result with (8), the expected value of the queue length at the beginning of a randomly chosen slot, we see that \( E'(1) \), \( D'(1) \) and \( V'(1) \) satisfy Little's theorem:

\[
D'(1) E'(1) = V'(1).
\]

This is not as obvious as it may seem, because \( D(z) \) does not describe the exact packet delay: in our calculations concerning \( D(z) \), we only counted the complete slots and neglected the fraction of the slot of arrival, that a packet stays in the station. This seems to imply that packet arrivals are registered by a station at the end of the slots. A more rigorous discussion of this phenomenon can be found in Brunel and Kim [8].

Rubin and Zhang [1] have derived a lower bound

\[
0 < \sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} \tag{18}
\]

and an upper bound

\[
\sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} < N-1
\]

for the complex sum in expression (8) for the expected value of the queue size at the beginning of a randomly chosen slot, that allow a quick approximation of this quantity. However, the above upper bound of the complex sum is not very useful to approximate the expected value of the packet delay in the full range of possible values of \( E'(1) \), as the corresponding upper bound for the mean packet delay goes to infinity for
small values of $E'(1)$. Therefore, we derive a second upper bound that does not have this inconvenience. Using (16), we can obtain the following expression:

$$
\sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} = \frac{2}{(N - (N + L) E'(1))} \cdot \sum_{k=1}^{N} \frac{(k-1) V_{L+k}(0)}{(N - 1)}.
$$

We now express that during a A-slot, the average number of departures exceeds the average number of arrivals:

$$1 - V_{L+k}(0) > E'(1), \quad 1 \leq k \leq N.
$$

This is motivated by the fact that on the average, in the steady state, there are more departures than arrivals of packets during a complete A-period. Using this relation in the above expression of the complex sum, we obtain the following upper bound:

$$\sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} < \frac{(N - 1) L E'(1)}{N - (N + L) E'(1)}.
$$

One can verify that this upper bound, and the one found by Rubin and Zhang [1], coincide in the point $E'(1) = N/(N + 2L)$. Therefore, we define the following upper bound for the complex sum:

$$\sum_{j=1}^{N-1} \frac{E(z_j) + z_j}{E(z_j) - z_j} \leq \frac{(N - 1) L E'(1)}{N - (N + L) E'(1)}, \quad 0 \leq E'(1) \leq \frac{N}{N + 2L}, \quad 1 \leq k \leq N.
$$

The inequalities (18) and (19) constitute a lower and upper bound for the complex sum, that make it possible to derive tight upper and lower bounds for the mean queue size at the beginning of a randomly chosen slot and the expected mean packet delay, for any value of $E'(1)$.

### 6. Memoryless arrival processes

We call an arrival process “memoryless”, if the number of arrivals during a slot with at least one arrival is geometrically distributed:

$$E(z) - E(0) = \frac{z}{1 - (1 + \theta)^{-1}}.
$$

This means that the most general form for the generating function describing a memoryless arrival stream is given by:

$$E(z) = \frac{z + (1 + \theta) E(0)(1 - z)}{1 + \theta(1 - z)}, \quad 0 \leq E(0) \leq 1, \quad 0 \leq \theta.
$$

Two well-known memoryless arrival processes, usually referred to as the Bernoulli and geometric arrival process, are special cases of (20a,b), namely for $E'(0) = 0$ and $\theta = 1 - E(0)/E(0)$ respectively.

It can be shown that eq. (14) is in accordance with the results found by Rubin and Zhang [1], who derived an expression for the generating function of the packet delay in the two special cases of a Bernoulli and a geometrical arrival stream. We give a proof for the case of a Bernoulli arrival process with probability generating function

$$E(z) = 1 - p + p z.
$$

Using this arrival process in the expression (14) for the generating function of the packet delay, we can write:

$$D(z) = \frac{z^N}{N(N + L)} \cdot \sum_{s=0}^{N-1} \sum_{m=0}^{N-1} (a^{-s} z^{m})^m \cdot \frac{\alpha^a z^{N + L}}{z^{N - (1 - p) - s a^z^{N + L}}} \cdot \sum_{k=1}^{N} (a^{s} z^{k})^{1 - k} V_{L+k}(0),
$$

where the summation over $m$ has been introduced to eliminate part of the denominator in (14). Defining

$$f(z) = \frac{z^N}{N(N + L)(z^{N - (1 - p)})} \cdot g(z) = \frac{p z^{N + L}}{(z^{N - (1 - p)})},
$$

we obtain, using the series expansion of the denominator:

$$D(z) = f(z) \sum_{s=0}^{N-1} \sum_{m=0}^{N-1} \sum_{k=0}^{N} \sum_{j=0}^{s} (a^{s+k+m} N + L - a^{s+k+m-1}) \cdot z^{L(a + m - 1)} g(z) V_{L+k}(0).
$$

The sum over $s$ is worked out with the use of the identity (11):

$$D(z) = f(z) \sum_{m=0}^{N} \sum_{k=1}^{N} \sum_{j=0}^{s} \sum_{n=s}^{\infty} \delta(n - j - k + m) - \delta(n - j - k + m + 1) \cdot z^{L(a - m - 1)} g(z) V_{L+k}(0).
$$

The Kronecker delta functions are nonzero only if their argument is zero. Due to the bounds on $j$, $k$ and $m$ in the above summations, this means that only for positive values of $n$, we will find nonzero terms. Furthermore, if $k > m$ in the first, respectively $k > m + 1$ in the second Kronecker delta function in the above expression, there are no nonzero terms if $n$ is zero. Working out the sum over $j$, this yields:

$$D(z) = N f(z) \sum_{k=1}^{N} \left\{ \sum_{m=0}^{\infty} \sum_{s=0}^{N+k+m} (z^{L(k + m - 1)} - g(z)) z^{L(k + m - 1)} - \sum_{m=0}^{k-1} \sum_{s=0}^{N} (g(z) z^{-L} )^{m-k} + \sum_{m=0}^{k-2} (g(z) z^{-L} )^{m-k+1} \right\} \cdot V_{L+k}(0).
$$
Using relation (5), we obtain after some calculations

\[
D(z) = \frac{z^N}{(N+L)(1-p)p} \left\{ \begin{array}{c}
\frac{z^N}{(1-p)^N} - \frac{p^N z^N}{z^{N(L+1)}} \\
\sum_{k=1}^{N} \frac{z^N(1-p)^N}{p z^{N}} V_{L+k}(0) + (N-(N+L)p) \end{array} \right\}.
\]

This can be reduced to

\[
D(z) = V\left(\frac{z-1}{p} + 1\right),
\]

with \( V(z) \) the generating function of the queue size at the beginning of a randomly chosen slot, given by expression (7). This is the result that was found by Rubin and Zhang [1]. This is, in fact, a special case of a much more general property. One can prove, as in Bruneel and Kim [8], that for all discrete-time queueing systems with one single server and a memoryless uncorrelated arrival stream, the following relation holds:

\[
D(E(z)) = V(z)
\]

with \( E(z) \), \( V(z) \) and \( D(z) \) the probability generating functions of the arrival process, the queue size at the beginning of a randomly chosen slot, and the packet delay respectively. It is clear that the above described system, namely a TDMA scheme with a Bernoulli arrival process, is a special case of such a discrete-time queueing system.

7. Numerical example

Let us consider a TDMA scheme where \( N = 3 \) and \( L = 8 \), and let us assume a geometrical arrival process, i.e.,

\[
E(z) = \frac{1}{1+\lambda - \lambda z}, \quad E'(1) = \lambda.
\]

In Fig. 5, the average packet delay is plotted versus the channel utilization \( \phi \), which is defined as

\[
\phi = \frac{N+L}{N} E'(1) = \frac{11}{3} \lambda.
\]

Also shown in this figure are the upper and lower bound of the mean packet delay, obtained by using the inequalities (19) and (18) in (17). The figure reveals that the upper and lower bound are sufficiently tight to allow a quick estimate of the average packet delay.

8. Conclusions

In this paper, we have derived an exact expression for the probability generating function of the packet delay in a TDMA scheme were multiple contiguous slots are assigned to a station, which is valid for a general uncorrelated arrival process. In doing so, we have generalized the work of Rubin and Zhang [1], where the same problem was solved only for geometric and Bernoulli arrival process, and the approximate analysis by Bruneel [2] valid for general independent arrivals under the assumption that \( N \ll N + L \). From our result it is possible to derive explicit expressions for the various moments of the packet delay. In particular, we have considered the mean packet delay, for which, in addition, tight lower and upper bounds were obtained.

References

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