Exact derivation of transient behavior for buffers with random output interruptions *

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Abstract


A simple model for discrete-time buffer systems with random output interruptions and general independent arrivals is considered. A method is developed for the derivation of transient statistics of this kind of queueing system. In particular, an exact algorithm is presented for the calculation of such quantities as the (time-dependent) probability of an empty buffer and the mean buffer occupancy. A few examples illustrate the analysis.

Keywords: discrete-time queueing systems, transient analysis, analytical solution, complex analysis.

1. Introduction

Discrete-time queueing models have been used for a number of years to analyze the (steady-state) queueing performance of various synchronous digital communication systems. Examples of such studies are given in references [1–5,8,11,12,14,18–20]. Recently, there has been a renewed interest in these models, in view of their applicability in the performance evaluation of broadband packet switches; see e.g. [9,13,15,17,21]. Finite [11,18,19] as well as infinite [1–5,7–9,11–15,17,20,21] capacity models have been analyzed. Various models have been adopted for the arrival process of the system. For instance, a Poisson arrival process was considered in [8,12,14,18], a compound Poisson process in [11], a mixture of Poisson and compound Poisson in [19], Bernoulli arrivals in [9,13,15,17,21], a binomial arrival process in [9,13,15,17], whereas general independent arrivals were chosen in [1–5,20]. On the other hand, the various studies also differ in the model used for the “output process” (or “service process”) of the systems. Both single-server [2–5,9,11–15,17–21] and multiple-server [1,8] systems have been investigated. Not only has the case been considered where the servers are permanently available [9,11,13,17,20,21], but considerable research effort has also been devoted to systems in which the servers are exposed to different kinds of interruptions [5,8,12,14,18,19], or, closely related to this, priority queueing models (e.g. [21]) or vacation queueing models (e.g. [7]).

Whereas there is a vast literature dealing with the steady-state analysis of discrete-time queueing systems, only few research efforts have addressed the derivation of transient statistics. Chu and Konheim [6] have described a formal method to derive the full time-dependent behavior of a system without service interruptions in the form of a transform function having two arguments (one for the queue length and one for the discrete time parameter). However their analysis yields no explicit results for such quantities as the probability of an empty system and the mean queue length (in terms of the discrete time index). Jenq [16] has developed an approximate algorithm to calculate the mean and the variance of the queue length, for a system with service interruptions. The aim of the present paper is to present a new analytical technique for the derivation of the exact time-dependent queueing behavior of discrete-time buffer

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systems. More specifically, we have chosen to analyze exactly the same model as in Jenq [16], because it is general enough to capture systems with server interruptions as well as systems without server interruptions (possibly with variable service times) and with general independent arrivals, and, on the other hand, simple enough to allow a clear presentation of our method of analysis.

The exact assumptions of the model are given in Section 2. The transient queueing analysis of the system is presented in Sections 3, 4 and 5. Several specific examples are discussed in detail in Section 6, and some final comments are given in Section 7.

2. The investigated system

The system under consideration is depicted in Fig. 1. It consists of a buffer for the storage of digital data, a number of input channels and one single output channel. The buffer is assumed to be of infinite capacity. Data units enter the system in a random fashion via the input lines and are taken out of the buffer via the output channel for transmission to their destination. We assume that the time axis is divided into discrete time-units, called slots. The output line is capable of transmitting exactly one data unit per slot. However, the transmission may be unsuccessful owing to the occurrence of random break downs of the output line, so that the data units are blocked and cannot leave the buffer.

We assume that both the arrivals and the service breakdowns occur independently from slot to slot. More specifically, we use the symbol $E(z)$ to indicate the probability generating function of the number of arrivals per slot and the symbol $\sigma$ to denote the probability of a successful transmission. We make no specific assumptions with respect to the shape of the arrival process, i.e., the function $E(z)$ is an arbitrary probability generating function.

3. Transient queueing analysis

Let $V_j(z)$ denote the probability generating function of the queue length at the beginning of the $(j+1)$-th slot. Then, by using standard $z$-transform techniques, it can be shown that, for all $j \geq 1$, $V_j(z)$ can be derived from $V_{j-1}(z)$ through the following relationship:

$$V_j(z) = \frac{E(z)}{z} \left\{ (1 - \sigma)z + \sigma \right\} V_{j-1}(z) + \sigma (z - 1)V_{j-1}(0),$$

where we have used the notations introduced in Section 2.

Next, let $V(x, z)$ be the transform function of the sequence $\{V_j(z)\}$ with respect to the discrete time parameter $j$, i.e.,

$$V(x, z) \triangleq \sum_{j=0}^{\infty} V_j(z) x^j.$$  \hspace{1cm} (2)

Then the following expression for $V(x, z)$ can be derived from equation (1) by multiplying both

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sides of the equation by \( x^j \) and summing over all \( j \geq 1 \):
\[
V(x, z) = \frac{zV_0(z) + \alpha x V(x, 0)(z - 1) E(z)}{z - x(\alpha + (1 - \sigma)z) E(z)}.
\]
(3)

The right hand side of eq. (3) is fully determined except for the unknown boundary term \( V(x, 0) \), which is the transform function of the sequence \( \{ V_j(0) \} \), i.e., \( V(x, 0) \) describes the evolution in time of the probability of an empty buffer. A method to determine \( V(x, 0) \), or, equivalently, the sequence \( \{ V_j(0) \} \), is discussed in the next sections. Once these quantities are known, the full transient behavior of the queue can be derived from eq. (3).

In particular, if \( N_j \) denotes the mean queue length at the beginning of the \((j + 1)\)-th slot, i.e.,
\[
N_j \triangleq \frac{dV_j(z)}{dz} \bigg|_{z=1},
\]
and \( N(x) \) is the transform function of the sequence \( \{ N_j \} \), i.e.,
\[
N(x) \triangleq \sum_{j=0}^{\infty} N_j x^j,
\]
then \( N(x) \) can be obtained as
\[
N(x) = \left. \frac{dV(x, z)}{dz} \right|_{z=1}.
\]
In view of equation (3), \( N(x) \) can thus be expressed as
\[
N(x) = \frac{N_0 + \alpha x V(x, 0)}{1 - x} - \frac{(\sigma - \lambda)x}{(1 - x)^2},
\]
(4)
where \( N_0 \) indicates the initial mean buffer occupancy and \( \lambda \) denotes the mean arrival rate per slot, i.e.,
\[
\lambda \triangleq \frac{dE(z)}{dz} \bigg|_{z=1}.
\]
From the definitions of \( V(x, 0) \) and \( N(x) \) it is easily seen that eq. (4) is equivalent to
\[
N_j = N_{j-1} + \lambda - \sigma + \sigma V_{j-1}(0),
\]
(5)
for all \( j \geq 1 \), which is also intuitively clear. Equation (5) can be used for a recursive determination of the time-dependent mean queue length \( N_j \), on condition that the sequence \( \{ V_j(0) \} \) is known.

In a similar manner higher moments of the buffer occupancy can be derived from eq. (3), by computing higher order derivatives of \( V(x, z) \) with respect to the argument \( z \).

4. Determination of \( V(x, 0) \)

In order to determine the function \( V(x, 0) \) we return to the eqs. (2) and (3).

From the definition in eq. (2) it follows that \( V(x, z) \) is an analytic function of \( z \) inside the unit disk \( \{ z : |z| \leq 1 \} \) of the \( z \)-plane, for all values of \( x \) inside \( \{ x : |x| < 1 \} \). However, it can be shown by means of Rouche’s theorem (see e.g. [10]) that the denominator of (3) has exactly one zero inside the unit disk of the \( z \)-plane for all \( |x| < 1 \). Let \( z' = f(x) \) denote this zero. Then \( f(x) \) must be a zero of the numerator of (3) as well in order to guarantee the analyticity of \( V(x, z) \) inside \( \{ z : |z| \leq 1 \} \). In general, this condition leads to the following expression for \( V(x, 0) \):
\[
V(x, 0) = \frac{\sigma + (1 - \sigma) f(x)}{\sigma [1 - f(x)]} V_0(f(x)), \quad |x| < 1.
\]
(6)

Remark: There are two special cases whereby \( V(x, 0) \) cannot be obtained from equation (6), namely the case \( \sigma = 0 \) and the case \( E(0) = 0 \). In these two (trivial) cases both the denominator and the numerator of (3) have a zero at \( z = 0 \), irrespective of the value of \( x \). An expression for \( V(x, 0) \) can then be obtained by taking the limit of (3) for \( z \rightarrow 0 \). In the sequel, we will assume \( \sigma \neq 0 \) and \( E(0) \neq 0 \).

It now remains for us to determine the zero \( z' = f(x) \) as an explicit function of \( x \) from the equation
\[
z - x[\sigma + (1 - \sigma) z] E(z) = 0.
\]
(7)
In general, this may be not such an easy task. However, for specific choices of the generating function \( E(z) \) of the arrival process the solution of eq. (7) is straightforward. We give two examples.

Example 1: Bernoulli arrivals

In this case the number of arrivals per slot is \( 1 \) or \( 0 \) with probabilities \( \lambda \) or \( (1 - \lambda) \) respectively. The generating function \( E(z) \) is linear in \( z \),
\[
E(z) = 1 - \lambda + \lambda z
\]
and (7) becomes a quadratic equation in \( z \), which has exactly one solution \( f(x) \) inside the unit disk of the \( z \)-plane, for all \( |x| < 1 \). This solution is given by

\[
f_B(x) = \frac{1 - \left[ \sigma \lambda + (1 - \sigma)(1 - \lambda) \right] x - \sqrt{D_B(x)}}{2(1 - \sigma) \lambda x},
\]

where

\[
D_B(x) = \left( 1 - \left[ \sigma \lambda + (1 - \sigma)(1 - \lambda) \right] x \right)^2 - 4\sigma(1 - \sigma)\lambda(1 - \lambda)x^2.
\]

(8)

(9)

**Example 2: geometric arrivals**

With this kind of arrival process, the probability of \( n \) data units entering the buffer in one slot, is given by

\[
\frac{1}{1 + \lambda} \left( \frac{\lambda}{1 + \lambda} \right)^n, \quad n \geq 0.
\]

The corresponding generating function is

\[
E(z) = \frac{1}{1 + \lambda - \lambda z},
\]

so that, once again, (7) reduces to a quadratic equation in \( z \), yielding the following solution:

\[
f_G(x) = \frac{1 + \lambda - (1 - \sigma)x - \sqrt{D_G(x)}}{2\lambda},
\]

where

\[
D_G(x) = \left[ 1 + \lambda - (1 - \sigma)x \right]^2 - 4\lambda\sigma x.
\]

(10)

(11)

In the two above examples \( f(x) \) and hence \( V(x, 0) \) can be determined as explicit functions of the variable \( x \). The sequence \( \{V_j(0)\} \) can then be obtained by inverting the expression for \( V(x, 0) \), which may be accomplished in the way described in Section 6, where we discuss some numerical examples. Another more general technique to arrive at the \( V_j(0) \)'s, even in cases where \( f(x) \) cannot be derived explicitly, is presented in Section 5.

5. **Direct determination of the \( V_j(0) \)'s**

The determination of \( z = f(x) \) as a solution (with respect to the unknown \( z \)) of eq. (7), for a given value of \( x \), may be quite difficult. However the inverse problem, i.e., determining \( x = r(z) \) as the solution (with respect to the unknown \( x \)) of (7), for a given value of \( z \), is straightforward, due to the fact that eq. (7) is linear in \( x \). Indeed, this solution is

\[
x = r(z) \triangleq \frac{z}{\sigma + (1 - \sigma)z} E(z).
\]

(12)

It is clear from eq. (12) that, if \( |z| < 1 \) is chosen sufficiently small, then also \( |x| = |r(z)| \) will be smaller than unity. This is true because we have assumed that \( \sigma \neq 0 \) and \( E(0) \neq 0 \), so that

\[
|z| \approx \frac{|z|}{\sigma E(0)}, \quad \text{if } |z| \ll 1.
\]

(13)

It follows that \( V(x, z) \) must be analytic for the choice \( (x, z) = (r(z), z) \), if \( |z| \) is chosen sufficiently small. Hence the numerator of (3) must vanish for \( x = r(z) \), which leads to the following result:

\[
V(r(z), 0) = V_0(z) \frac{\sigma + (1 - \sigma)z}{\sigma(1 - z)}, \quad \text{for } |z| \ll 1.
\]

(14)

Instead of what we would really want—an expression for \( V(x, 0) \)—eq. (14) gives us \( V(r(z), 0) \) as an explicit function of \( z \) for small \( |z| \). We will now show that this is enough to derive the quantities \( V_j(0) \).

Indeed, from the definition of \( V(x, 0) \) it follows that \( V_j(0) \) is the coefficient of \( x^j \) in the expansion of \( V(x, 0) \) about \( x = 0 \). Hence, in terms of complex analysis, \( V_j(0) \) can be obtained as the residue of the function \( x^{-1-j}V(x, 0) \) in the point \( x = 0 \). This residue can be expressed in integral form as follows:

\[
V_j(0) = \frac{1}{2\pi i} \oint_{C_0} V(x, 0)x^{-1-j} \, dx,
\]

(15)

where \( i = \sqrt{-1} \) and the integral is evaluated in the complex \( x \)-plane around a small closed contour \( C_0 \) which surrounds the origin \( x = 0 \), but no poles of \( V(x, 0) \).

Our next step is to introduce in (15) a change of variable indicated by the substitution \( x = r(z) \), where \( r(.) \) is the function defined in (12) and \( z' \) is the zero inside \( \{ z : |z| \leq 1 \} \) of the denominator of (3), mentioned above. From the discussion in Sec-
tion 4 it follows that this is a valid substitution for all \( |x| < 1 \), because each such \( x \) gives rise to one single \( z' \) and vice versa. The contour \( C_0 \) in the \( x \)-plane is transformed into an equivalent contour \( C_1 \) in the \( z' \)-plane, which encircles the point \( z' = 0 \) exactly one time. As a result we obtain (writing \( z \) instead of \( z' \) again, for convenience)

\[
V_j(0) = \frac{1}{2\pi i} \oint_{C_1} V(r(z), 0) [r(z)]^{-1-j} r'(z) \, dz,
\]

where \( r'(z) \) denotes the first derivative of \( r(z) \) with respect to \( z \), which is given by

\[
r'(z) = \frac{\sigma E(z) - z [\sigma + (1 - \sigma) z] E'(z)}{[\sigma + (1 - \sigma) z]^2 [E(z)]^2}.
\]

The behavior of \( r(z) \) in the vicinity of \( z = 0 \) (eq. (13)) makes clear that \( |z| \ll 1 \) on the contour \( C_1 \), so that the expression (14) for \( V(r(z), 0) \) is valid for all \( z \) on \( C_1 \). It follows that \( V_j(0) \) can be expressed as

\[
V_j(0) = \frac{1}{2\pi i} \oint_{C_1} V_0(z) \left\{ \frac{[\sigma + (1 - \sigma) z]^{j} [E(z)]^{j-1}}{\sigma (1 - z) z^{j+1}} \times \left\{ \sigma E(z) - z [\sigma + (1 - \sigma) z] E'(z) \right\} \right\} \, dz,
\]

which contains known quantities only.

Finally, eq. (16) can be interpreted as follows: \( V_j(0) \) is the residue of the integrand of the integral in (16), in the point \( z = 0 \). In other words: \( V_j(0) \) can be obtained as the coefficient of \( z^j \) in the expansion of

\[
P_j(z) \triangleq V_0(z) \left[ \sigma + (1 - \sigma) z \right]^j [E(z)]^{j-1} \times \frac{\sigma E(z) - z [\sigma + (1 - \sigma) z] E'(z)}{\sigma (1 - z)}
\]

about the point \( z = 0 \). Now it is clear that the function \( P_j(z) \) defined in (17) is analytic for all \( |z| < 1 \), because \( V_0(z) \), \( E(z) \) and \( E'(z) \) are. Hence, \( P_j(z) \) has a Taylor expansion about \( z = 0 \),

\[
P_j(z) = \sum_{k=0}^{\infty} p_j(k) z^k.
\]

The quantity \( V_j(0) \) can thus be obtained as

\[
V_j(0) = p_j(j) = \left. \frac{d^j P_j(z)}{dz^j} \right|_{z=0}.
\]

Equation (19) gives a formal expression for the quantity \( V_j(0) \), albeit not in a very manageable form. A more practical way of determining \( V_j(0) \) is to derive the expansion (18) for \( P_j(z) \) by means of any method—which may depend on the particular form of \( V_0(z) \) and \( E(z) \)—and to select the coefficient \( p_j(j) \). A practical example of this approach is given in the next section.

6. Numerical examples

We consider two important particular cases for the arrival process, i.e., geometric arrivals and Poisson arrivals. In both cases we assume that the buffer is empty at the beginning of the first slot, i.e.,

\[
V_0(z) = 1, \quad \text{for all } z.
\]

6.1. Geometric arrivals

In this case an explicit expression for \( V(x, 0) \) is available, as we have mentioned in Section 4. In view of (20), eqs. (6), (10) and (11) yield

\[
V(x, 0) = \frac{2\sigma - (1 + \lambda) + (1 - \sigma) x + H(x)}{2\sigma (1 - x)},
\]

where \( H(x) \) is defined as

\[
H(x) \triangleq \sqrt{(1 + \lambda - x + \sigma x)^2 - 4\lambda \sigma x}.
\]

The determination of the \( V_j(0) \)'s from (21) and (22) can be accomplished as follows.

In a first step, we derive a series expansion for \( H(x) \) about the point \( x = 0 \):

\[
H(x) = \sum_{j=0}^{\infty} h(j) x^j.
\]

This can be done as follows. By taking the logarithmic derivative of (22) with respect to \( x \), we obtain

\[
\frac{H'(x)}{H(x)} = \left\{ (1 - \sigma)^2 x - [1 - \sigma + \lambda (1 + \sigma)] \right\} \times \left\{ (1 - \sigma)^2 x^2 - 2[1 - \sigma + \lambda (1 + \sigma)] x + (1 + \lambda)^2 \right\}^{-1}
\]
or, equivalently,
\[
\left((1 - \sigma)^2 x^2 - 2[1 - \sigma + \lambda (1 + \sigma)] x + (1 + \lambda)^2\right) \\
\times \sum_{j=1}^{\infty} j h(j) x^{j-1}
\]
\[
= \left((1 - \sigma)^2 x - [1 - \sigma + \lambda (1 + \sigma)]\right) \sum_{j=0}^{\infty} h(j) x^{j},
\]
where we have replaced $H(x)$ and $H'(x)$ by their respective series expansions. Identification of equal powers of $x$ in both sides of (23) then leads to
\[
h(1) = \frac{1 - \sigma + \lambda (1 + \sigma)}{(1 + \lambda)^2} h(0)
\]
(24)
and
\[
h(j) = \left[-(j - 3)(1 - \sigma)^2 h(j - 2)
\right.
\]
\[
+ (2j - 3)[1 - \sigma + \lambda (1 + \sigma)] h(j - 1)\]
\[
/j(1 + \lambda)^2
\]
(25)
for \( j \geq 2 \). The quantities \( h(j) \) can now be calculated recursively from (24) and (25), keeping in mind that the initial value \( h(0) \) is given by

\[
h(0) = H(x)|_{x=0} = 1 + \lambda.
\]

Once the quantities \( h(j) \) are known for all \( j \), a recursive formula for the \( V_j(0) \)'s can be obtained from (21). Indeed, (21) can be written as

\[
2\sigma(1 - x)V(x, 0) = 2\sigma - (1 + \lambda) + (1 - \sigma)x + H(x)
\]
or, equivalently,

\[
2\sigma(1 - x) \sum_{j=0}^{\infty} V_j(0) x^j
\]

\[
= 2\sigma - (1 + \lambda) + (1 - \sigma)x + \sum_{j=0}^{\infty} h(j) x^j.
\]

Identifying, once again, equal powers of \( x \), we finally obtain the following result:

\[
V_0(0) = 1,
\]

\[
V_1(0) = 1/(1 + \lambda),
\]

\[
V_j(0) = V_{j-1}(0) + h(j)/2\sigma, \quad j \geq 2.
\]

The algorithm described above can be implemented very easily on a digital computer. Some results are represented in Figs. 2 and 3, where the time-dependent probability of an empty buffer \( (V_j(0)) \) and the mean buffer occupancy \( (N_j) \) are plotted versus the discrete time index \( j \), for a value \( \sigma = 0.5 \) and various values of the arrival intensity \( \lambda \). Notice that the \( N_j \)'s are calculated recursively from the \( V_j(0) \)'s by means of equation (5). For \( \lambda < \sigma \), the curves clearly show the convergence of \( V_j(0) \) and \( N_j \) to their equilibrium values, as \( j \) increases:

\[
V_j(0) \rightarrow V(0) = 1 - \lambda/\sigma,
\]

\[
N_j \rightarrow N = \lambda/(\sigma - \lambda).
\]

We note, in passing, that the time required for the system to reach its steady state decreases as the quantity \( \sigma - \lambda \) increases. On the other hand, for \( \lambda \geq \sigma \), no steady state exists and

\[
V_j(0) \rightarrow 0
\]

\[
N_j \rightarrow \infty.
\]

6.2. Poisson arrivals

In this example we assume that the probability of \( n \) data units arriving at the buffer during one slot is given by the Poisson distribution

\[
e^{-\lambda n}n!, \quad n \geq 0,
\]

with corresponding generating function

\[
E(z) = e^{\lambda(z-1)}.
\]

Equation (7) is now a transcendental equation with respect to \( z \), for which an explicit solution \( f(x) \) cannot be obtained easily. We therefore resort to the approach discussed in Section 5.

In view of eqs. (20) and (26), the function \( P_j(z) \) defined in (17) is given by

\[
P_j(z) = P_0(z) Q_j(z)
\]

where

\[
P_0(z) = (\sigma - \lambda z[\sigma + (1 - \sigma)z])/(\sigma(1 - z))
\]

and

\[
Q_j(z) = [\sigma + (1 - \sigma)z]^{-j} e^{\lambda(z-1)}.
\]

Let \( p_0(n) \) and \( q_j(n) \) denote the coefficients of \( z^n \) in the series expansions about \( z = 0 \) of \( P_0(z) \) and \( Q_j(z) \) respectively. Then from the derivations in Section 5 and eq. (27) it is clear that the quantity \( V_j(0) \) can be computed as

\[
V_j(0) = \sum_{n=0}^{j} p_0(n) q_j(j - n).
\]

It thus remains for us to determine the quantities \( p_0(n) \) for all \( n \) and \( q_j(n) \) for \( 0 \leq n \leq j \).

It is easily seen that the \( p_0(n) \)'s are given by

\[
p_0(n) = \begin{cases} 1 & \text{for } n = 0 \\ 1 - \lambda & \text{for } n = 1 \\ 1 - \lambda/(\sigma - \lambda) & \text{for } n \geq 2. \end{cases}
\]

On the other hand, an easy way to arrive at the \( q_j(n) \)'s is as follows. First, we observe that

\[
q_j(0) = Q_j(z)|_{z=0} = \sigma^j e^{-\lambda \sigma}.
\]

Next, we take the logarithmic derivative of (29) with respect to \( z \), thus obtaining

\[
\frac{Q_j'(z)}{Q_j(z)} = j[1 - \sigma + \lambda\sigma + \lambda(1 - \sigma)z] \frac{1}{\sigma + (1 - \sigma)z}.
\]
which is equivalent to

\[ \sigma + (1 - \sigma)z \sum_{n=1}^{\infty} nq_j(n)z^{n-1} \]

\[ = j[1 - \sigma + \lambda \sigma + \lambda (1 - \sigma)z] \sum_{n=0}^{\infty} q_j(n)z^n. \]

Identifying equal powers of \( z \) in both sides of this identity then leads to

\[ q_j(1) = \frac{1 - \sigma + \lambda \sigma}{\sigma} q_j(0) \]

(33)

and

\[ q_j(n) = \frac{(j - n + 1)(1 - \sigma) + j\lambda \sigma}{n\sigma} q_j(n - 1) \]

\[ + \frac{j\lambda (1 - \sigma)}{n\sigma} q_j(n - 2) \quad \text{for } n \geq 2. \]

(34)

Equations (32)–(34) allow a recursive calculation of \( q_j(n) \) for all \( n \). Together with eqs. (30) and (31) they constitute an algorithm for the determination of \( V_j(0) \), and hence, by use of eq. (5), of \( N_j \), for all \( j \). We illustrate the analysis by means of a numerical example whereby \( \sigma = 0.7 \) and \( \lambda \) assumes values between 0.5 and 0.9. The transient mean buffer occupancy \( N_j \) for this choice of parameters is plotted versus the discrete time parameter \( j \) in Fig. 4. For \( \lambda < \sigma \) the curves converge to their equilibrium value, which is given by (see Hsu [5])

\[ N = \frac{\lambda(2 - \lambda)}{2(\sigma - \lambda)}, \]

whereas for \( \lambda \geq \sigma \) they tend to infinity. We observe, once again, that the system converges towards the steady state more rapidly as the quantity \( \sigma - \lambda \) increases. Although this requires further research, we believe this to be true regardless of the initial state of the buffer.

7. Concluding remarks

In this paper we have presented a method to derive the transient behavior of discrete-time buffer systems with random output interruptions and general independent arrivals. More specifically, we have developed an exact algorithm for the calculation of the most important performance measures of such a buffer system as functions of the discrete time parameter. This algorithm, which comes down to determining the coefficients of the series expansion about the origin of some known function, has a comparable complexity as the approximate algorithm proposed by Jenq [16] for the same problem.

A comparison of the results obtained for \( N_j \) with both methods was carried out. More specifically, the curve for \( \lambda = 0.6 \) in Fig. 4 was compared with the curve in Fig. 2 of Jenq’s paper [16] for the same system. The comparison revealed that Jenq’s approximate algorithm yields results which are quite close to the exact values obtained with our method.
The techniques proposed in this paper are based on the use of z-transforms and generating functions. This approach leads to (simple) algorithms for the determination of the time-dependent moments of the queue length, which do not require the derivation of the (whole) queue-length distribution at each discrete time-point, as would be the case if the problem were solved in the time domain.

Our analysis has been restricted to infinite-capacity queues. The author expects that an extension to finite-waiting-room models will be far from obvious, because these models do not lend themselves easily to solutions in terms of z-transforms. On the other hand, more general types of server interruption processes than the one considered here, e.g., first-order Markovian or even more general renewal-type interruptions (as described, for instance, in [2–5]), are probably amenable to the same type of analysis as the one described in this paper. Obviously, the resulting algorithms will be of a more complex nature, as more complicated state descriptions (than just the queue length itself) are involved in such systems. Future research will consider these generalizations.

References