A general model for the behaviour of infinite buffers with periodic service opportunities *

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In this paper a discrete buffered system with infinite buffer size is considered, where the output channels are available only periodically. The analysis holds for a general bulk arrival process and a general probability mass function for the number of output channels available. The condition for stochastic equilibrium is obtained and the generating function of the buffer occupancy is derived. Various examples illustrate the generality of the analysis and several previously found results are obtained as special cases of it.

1. Introduction

Many authors have dealt with the problem of analyzing the behaviour of buffers in various circumstances [1–5]. The system they consider has the following general form (Fig. 1): Customers (messages, characters,…) arrive in a stochastic manner via a number of input channels, wait in the buffer for some time and are then taken out via one of the output channels. The output channels (transmission lines) are assumed to transmit data with constant speed and thus the time to transmit each message or character of fixed length is constant, i.e., the service time of the customers is constant. Furthermore, the data transmission is generally assumed to be synchronous, i.e., the data is taken out synchronously from the buffer for transmission at each discrete clock time. Messages arriving between two consecutive clock times have to wait until the next clock time to begin transmission even if the buffer is idle at the time of arrival. Most analyses assume a Poisson arrival process and a fixed number of output channels which may possibly be blocked for a constant fraction of time. In this paper however the analysis is made for a general arrival process (Poisson or not Poisson, bulk arrivals or single arrivals), and a stochastic number of available output channels during each clock time period is used in order to model the possible blocking of (some of) the output channels. The buffer is assumed to have an infinite waiting room.

As a result the probability generating function of the buffer occupancy is obtained and various moments of the buffer occupancy may be computed. Several examples illustrate the generality of the analysis, and the results which were derived previously in [3] and [5] are found as special cases.

2. General assumptions and terminology

The clock time period is defined as the time required to service one message, i.e., during one clock time period each of the (available) output channels is capable of transmitting exactly one message. The total number of arrivals during the $k$th unit service interval is denoted by $a_k$. The $a_k$’s are assumed to be i.i.d. with probability mass function

$$ a(n) \triangleq \text{Prob}[n \text{ messages arrive during the } k \text{ th unit service interval}]$$

$$ = \text{Prob}[a_k = n], \quad n = 0, 1, 2, \ldots .$$

The number of output channels is denoted by $m$. The number of output channels available during the $k$th unit service interval is assumed to be a random variable $c_k$. The $c_k$’s are also assumed to be i.i.d. according to a probability mass function

$$ c(j) \triangleq \text{Prob}[j \text{ output channels are available during the } k \text{ th unit service interval}]$$

$$ = \text{Prob}[c_k = j], \quad j = 0, 1, 2, \ldots , m.$$
Further we make the assumption that during a unit service interval no arrivals can occur before the possible departures have happened. Therefore, when the buffer contains \( n \) messages at the beginning of a unit service interval, no other messages than these \( n \) can leave the buffer during this particular unit service interval.

### 3. Derivation of the probability generating function of the buffer occupancy

Let us define the following random variables:

\[ v_k = \text{buffer occupancy after the } k \text{th unit service interval}, \]
\[ v_{k+1} = \text{buffer occupancy after the } (k+1)\text{th unit service interval}, \]

and derive the basic state equations.

Two cases have to be considered.

If, after the \( k \)th clock time interval there are more output channels available than the number of messages in the buffer, all these messages are transmitted, and the only messages in the buffer after the \((k+1)\)th clock time interval are the ones that arrived during that interval. Formally,

\[ v_{k+1} = a_{k+1}. \tag{1} \]

If, however, the buffer occupancy after the \( k \)th clock time interval is greater than or equal to the number of output channels available during the \((k+1)\)th clock time interval \( c_{k+1} \), there are no more than \( c_{k+1} \) departures in this interval, and we may write formally:

\[ v_{k+1} = v_k - c_{k+1} + a_{k+1}. \tag{2} \]

Equations (1) and (2) can be combined into a single equation as

\[ v_{k+1} = a_{k+1} + (v_k - c_{k+1}) \cdot U(v_k - c_{k+1}) \tag{3} \]

where the function \( U(x) \) is defined as

\[ U(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \]

Equation (3) will now be used in order to find the probability generating function of the equilibrium buffer occupancy \( V(z) \), where \( V(z) \) is defined as

\[ V(z) = \lim_{k \to \infty} E[v^{(k+1)}] = E[v^{a+c-(c-1)\cdot U(c-1)}] \tag{4} \]

where \( a \) and \( c \) are random variables with the same probability mass functions as \( a_{k+1} \) and \( c_{k+1} \) respectively, which are indeed independent of \( k \).

Since \( v_k, a_{k+1} \) and \( c_{k+1} \) are mutually independent random variables, the same holds for \( v, a \) and \( c \) in (4), which can thus be written as

\[ V(z) = E[v^a] \cdot E[z^{(c-1)\cdot U(c-1)}] = A(z) \cdot G(z). \tag{5} \]

Here \( A(z) \) is the Z-transform of \( a(n) \), defined in the previous section, i.e.,

\[ A(z) \triangleq Z\{a(n)\} (z), \]

and

\[ G(z) = E[z^{(c-1)\cdot U(c-1)}] = E_i \left[ E_i \left[ z^{(c-1)\cdot U(c-1)} \right] \right] = E_i \left[ \sum_{i=0}^{c} p(i) + \sum_{i=c+1}^{\infty} p(i)z^{-i} \right], \]

where \( p(i) \triangleq \text{Prob}(v = i) \).

\[ G(z) = E_i \left[ \sum_{i=0}^{c} p(i) + z^{-c} \left( V(z) - \sum_{i=0}^{c} p(i)z^i \right) \right] = E_i \left[ V(z)z^{-c} + \sum_{i=0}^{c} p(i)[1 - z^{i-c}] \right]. \]

We define \( C(z) \) as the Z-transform of \( c(n) \), defined in the previous section, i.e.,

\[ C(z) \triangleq Z\{c(n)\} (z). \]

Then,

\[ G(z) = V(z) \cdot C\left(\frac{1}{z}\right) + H(z), \tag{6} \]

where

\[ H(z) = E_i \left[ \sum_{i=0}^{c} p(i)[1 - z^{i-c}] \right] = \sum_{j=0}^{m} c(j) \sum_{i=0}^{j} p(i) \cdot [1 - z^{i-j}] = \sum_{i=0}^{m} p(i) \sum_{j=0}^{m} c(j) \cdot [1 - z^{i-j}] \]

\[ \triangleq z^{-m} \sum_{i=0}^{m} p(i)\alpha_i(z). \tag{7} \]
Here the \( \alpha_i(z) \)'s are polynomials of degree \( m \) defined by

\[
\alpha_i(z) \triangleq \sum_{j=i}^{m} c(j) [z^m - z^{m+i-j}].
\]  
(8)

Since \( \alpha_m(z) = 0 \), the summation in (7) goes from \( i = 0 \) to \( i = m-1 \) and from (5), (6) and (7) we obtain

\[
V(z) = A(z)
\cdot \left( V(z) \cdot C \left( \frac{1}{z} \right) + z^{-m} \sum_{i=0}^{m-1} p(i) \alpha_i(z) \right).
\]

\( V(z) \) is then obtained as

\[
V(z) = \frac{\sum_{i=0}^{m-1} p(i) \alpha_i(z)}{z^m - z^m \cdot C \left( \frac{1}{z} \right) \cdot A(z)}
\]  
(9)

where the \( \alpha_i(z) \)'s are given by (8).

Equation (9) gives the probability generating function of the buffer occupancy as a function of known quantities on one hand and the unknown probabilities \( p(0), p(1), \ldots, p(m-1) \) on the other hand.

In order for a stochastic equilibrium to exist, it is clear that the average number of message arriving in a clock time interval must be lower than the average number of output channels available during each clock time interval. Formally this means that a stochastic equilibrium is possible if and only if

\[
\bar{a} < \bar{c}
\]  
(10)

where \( \bar{a} \triangleq \sum_{n=0}^{\infty} na(n) \) and \( \bar{c} \triangleq \sum_{j=0}^{m-1} jc(j) \) are the expected values of the random variables \( a \) and \( c \) respectively.

In Appendix A it is shown that under Condition (10) the denominator of \( V(z) \), i.e.,

\[
E(z) \triangleq z^m - z^m \cdot C \left( \frac{1}{z} \right) \cdot A(z)
\]

has exactly \( m \) zeros \( z_0, z_1, \ldots, z_{m-1} \) in the unit circle \( \{ z : |z| < 1 \} \) of the complex plane, one of which equals 1, i.e., \( z_0 = 1 \).

However, these \( m \) zeros of \( E(z) \) must be zeros of the numerator of \( V(z) \) as well since \( V(z) \) is an analytic function of \( z \) in the unit circle \([6]\) whenever Condition (10) is fulfilled. This leads to \((m-1)\) linear equations in the \( p(i) \)'s (no equation is obtained for \( z_0 = 1 \)), which together with the condition \( V(1) = 1 \) yield the values of the unknown \( p(0), p(1), \ldots, p(m-1) \). Using this procedure the probability generating function \( V(z) \) of the buffer occupancy, as given by (9) can thus be completely specified.

We will now take a look at some special input and output processes.

4. Some particular distributions for the random variable \( a \)

Consider the case where messages arrive in bulks rather than separately. Let

\( s \) = the number of arrivals in a unit service time,
\( b_i \) = the number of messages in the \( i \)th bulk;

then

\[
a = \sum_{i=1}^{s} b_i.
\]

The \( b_i \) are assumed to be i.i.d. according to the probability mass function \( b(n) \), with \( Z \)-transform, i.e., probability generating function, \( B(z) \). The random variable \( s \) is distributed according to the mass function \( s(n) \) and probability generating function \( S(z) \).

Then

\[
A(z) \triangleq E \left[ z^a \right]
\]

\[
= E \left[ z^{\Sigma_{i=1}^{s} b_i} \right]
\]

\[
= E \left[ (B(z))^s \right]
\]

\( = S(B(z)) \).

(11)

The model with bulk arrivals is in many cases more realistic than the one with single arrivals, because in many data communication systems the input traffic is in bursts (strings of characters) rather than in single characters.

Moreover the case where messages arrive separately is a special case of the bulk arrival case with the bulk size equal to 1 in a deterministic manner, i.e., with \( B(z) = z \).
Another special case of particular interest [2] is the one where the bulk size is geometrically distributed, i.e., where
\[ b(n) = a(1 - a)^{n-1}, \quad n = 1, 2, \ldots, \quad 0 < a < 1 \]
and
\[ B(z) = \frac{az}{1 - (1 - a)z}. \quad (12) \]
Here \(1/a\) is the average bulk size.

Two interesting distributions for the random variable \( s \) are
(i) the Poisson distribution:
\[ s(n) = e^{-\lambda \frac{\lambda^n}{n!}}, \quad n = 0, 1, 2, \ldots \]
and
\[ S(z) = e^{-\lambda(1-z)}; \quad (13) \]
(ii) the geometric distribution:
\[ s(n) = \frac{1}{1+\lambda} \left( \frac{\lambda}{1+\lambda} \right)^n, \quad n = 0, 1, 2, \ldots \]
and
\[ S(z) = \frac{1}{1+\lambda(1-z)}. \quad (14) \]

Here \(\lambda\) is a parameter which equals the expected value of \( s \) in both cases.

The resulting arrival distributions are
- single Poisson arrivals:
\[ A_s(z) = e^{-\lambda(1-z)}; \quad (15) \]
- single geometric arrivals:
\[ A_3(z) = \frac{1}{1+\lambda(1-z)}; \quad (16) \]
- geometric bulk Poisson arrivals:
\[ A_4(z) = e^{-\lambda(1-z)(1-(1-a)z)}; \quad (17) \]
- geometric bulk geometric arrivals:
\[ A_4(z) = \frac{1-(1-a)z}{(1+\lambda)-(1+\lambda-a)z}. \quad (18) \]

It is easily seen that for \(a = 1\), \( A_3(z) \) and \( A_4(z) \) reduce to \( A_s(z) \) and \( A_3(z) \) respectively. Thus the cases \( A(z) = A_s(z) \) and \( A(z) = A_3(z) \) need not be treated separately.

5. Some particular distributions for the random variable \( c \)

Three special cases will be considered:
(i) There are always \( m \) output channels variable, i.e., \( c = m \) (deterministic):
\[ c(i) = \begin{cases} 0 & \text{if } i \neq m, \\ 1 & \text{if } i = m; \end{cases} \]
or
\[ C(z) = C_1(z) \triangleq z^m. \quad (19) \]
(ii) The \( m \) output channels are separated from the buffer by a gate which is open with probability \( 1 - \sigma \) and closed with probability \( \sigma \) (Fig. 2). In this case
\[ c = m \cdot D \]
where \( D \) is a random variable which equals 0 with probability \( 1 - \sigma \) and which equals 1 with probability \( \sigma \), whose probability generating function is thus
\[ D(z) = (1 - \sigma) + \sigma z. \]
Hence
\[ C(z) = C_2(z) \triangleq D(z^m) = (1 - \sigma) + \sigma z^m. \quad (20) \]
(iii) Each of the \( m \) channels is controlled by a gate as describe above (Fig. 3). In this case
\[ c = \sum_{i=1}^{m} D_i \]

Fig. 2. Buffer system where either all or no output channels are blocked.

Fig. 3. Buffer system where the output channels are blocked or not blocked independently.
where the $D_i$ are i.i.d. with the same probability mass function as $D$ above.

Then

$$C(z) = C_2(z) = E \left[ z^{\sum_{i=1}^{m} D_i} \right]
= [D(z)]^m
= [(1 - \sigma) + \sigma z]^m. \tag{21}$$

It is easily seen that $C_2(z)$ and $C_3(z)$ reduce to $C_1(z)$ for $\sigma = 1$. The case $C(z) = C_1(z)$ need hence not be treated separately.

6. The case where all the output channels are controlled by the same gate

In this case $C(z) = C_2(z)$, as defined in (20). From (8) and (9) it follows that

$$V(z) = \frac{A(z) \cdot \sigma \cdot \sum_{i=0}^{m-1} p(i) \cdot (z^m - z^i)}{z^m - \left[ \sigma + (1 - \sigma) z^m \right] \cdot A(z)}. \tag{22}$$

We now consider the two different input processes we mentioned earlier.

(a) Poisson arrivals in geometric bulks:

$$A(z) = A_s(z) = e^{-\lambda (1 - z)/(1 - (1 - \alpha) z)};$$

$$V(z) = \frac{\sigma \sum_{i=0}^{m-1} p(i) \cdot (z^m - z^i)}{z^m \left[ e^{\lambda (1 - z)/(1 - (1 - \alpha) z)} + \sigma - 1 \right] - \sigma}. \tag{23}$$

Let $z_0 = 1$, $z_1, \ldots, z_{m-1}$ be the $m$ zeroes of the denominator of this expression outside the unit circle in the complex plane, then $V(z)$ for this case can be written as

$$V(z) = V_s(z) \triangleq \frac{m \sigma - \lambda / \alpha}{\prod_{i=1}^{m-1} (1 - z_i)} \prod_{i=0}^{m-1} (z - z_i) \frac{z^m \left[ e^{\lambda (1 - z)/(1 - (1 - \alpha) z)} + \sigma - 1 \right] - \sigma}{z^m \left[ e^{\lambda (1 - z)/(1 - (1 - \alpha) z)} + \sigma - 1 \right] - \sigma}. \tag{24}$$

For $\alpha = 1$ we are in the single Poisson arrivals case. It is easily seen that for $\alpha = 1$ our results reduce exactly to the ones found by Georganas [3]. If, in addition, we let $m = 1$, we obtain the results found by Hsu [5].

(b) Geometric arrivals in geometric bulks:

$$A(z) = A_g(z) = \frac{1 - (1 - \alpha) z}{(1 + \lambda) - (1 + \lambda - \alpha) z};$$

$$V(z) = \frac{[1 - (1 - \alpha) z] \sigma \cdot \sum_{i=0}^{m-1} p(i)(z^m - z^i)}{z^m \left[ \sigma + \lambda - [\lambda + \sigma (1 - \alpha)] z \right] - \sigma \left[ 1 - (1 - \alpha) z \right]}. \tag{25}$$

Let $z^*$ be the unique zero of the denominator of this expression outside the unit circle in the complex plane, then $V(z)$ for this case can be written as

$$V(z) = V_g(z) \triangleq \frac{1 - z^* - (1 - \alpha) z}{\alpha \cdot \frac{z^* - z^*}{z^*}}. \tag{26}$$

The corresponding expected buffer occupancy is given by

$$\bar{N}_4 = \frac{dV_g(z)}{dz} \bigg|_{z=1} = \frac{1}{z^* - 1} - \frac{1 - \alpha}{\alpha}. \tag{27}$$

The results for the single geometric arrivals case can be derived from the above expressions by substituting the value 1 for the parameter $\alpha$.

7. The case where each output channel is controlled by its own gate

In this case $C(z) = C_s(z)$, as defined in (21). From (8) and (9) we find

$$A(z) = \sum_{i=0}^{m-1} p(i) \alpha_i(z);$$

$$V(z) = \frac{\sigma \cdot \sum_{i=0}^{m-1} \rho(i) \alpha_i(z)}{z^m - (\sigma + (1 - \sigma) z)^m \cdot A(z)}, \tag{28}$$

with the $\alpha_i(z)$'s given by

$$\alpha_i(z) = \sum_{j=0}^{m} \sigma_j \alpha_i^{(1 - \alpha)} z^{m+j}. \tag{29}$$

We consider the same input processes as before.
(a) Poisson arrivals in geometric bulks:

\[ A(z) = A_1(z) = e^{-\lambda(1-z)/(1-(1-\alpha)z)}; \]

\[ V(z) = V_1(z) \]

\[ = \frac{m(\sigma - \lambda/\alpha)}{m-1} \prod_{i=1}^{m-1} (1-z_i) \]

\[ \times \frac{\prod_{i=0}^{m-1} (z-z_i)}{z^m e^{\lambda(1-z)/(1-(1-\alpha)z)} - [\sigma + (1-\sigma)z]^m}. \]  

(29)

where \( z_0 = 1, z_1, \ldots, z_{m-1} \) are the \( m \) zeroes of the denominator of this expression inside the unit circle in the complex plane. The corresponding expected buffer occupancy is given by

\[ \overline{N}_e = \frac{dV(z)}{dz} \bigg|_{z=1} \]

\[ = \frac{\lambda(2-\lambda)/\alpha^2 - (m-1)[m\sigma(2-\sigma) - 2\lambda/\alpha]}{2(m\sigma - \lambda/\alpha)} \]

\[ + \sum_{i=1}^{m-1} \frac{1}{1-z_i}. \]  

(30)

As before the results for the case of single Poisson arrivals are obtained by setting \( \alpha = 1 \) in the above equations.

(b) Geometric arrivals in geometric bulks:

\[ A(z) = A_2(z) = \frac{1 - (1-\alpha)z}{(1+\lambda) - (1+\lambda-\alpha)z}; \]

\[ V(z) = V_2(z) \]

\[ = 1 - \frac{z^* - 1 - (1-\alpha)z}{z - z^*}. \]  

(31)

Here \( z^* \) is the unique zero of

\[ [(1+\lambda) - (1+\lambda-\alpha)z] z^m \]

\[ - [\sigma + (1-\sigma)z] z^m [1 - (1-\alpha)z] \]

outside the unit circle in the complex plane. The corresponding expression for the expected buffer occupancy reads

\[ \overline{N}_e = \frac{dV(z)}{dz} \bigg|_{z=1} = \frac{1}{z^* - 1} - \frac{1 - \alpha}{\alpha}. \]  

(32)

Again, putting \( \alpha = 1 \) in these equations leads to the expressions for the case of single geometric arrivals.

8. Examples

Example 1. One single output channel. In this case \( m = 1 \) and

\( C_1(z) = C_1(z) = 1 - \sigma + \alpha z \)

and thus the analyses of Sections 6 and 7 yield the same results. Furthermore, all the results can be given as explicit functions of the parameters of the system since the denominator of \( V(z) \) has only one zero inside the unit circle in the complex plane: \( z_0 = 1 \).

The following results are found:

(a) Poisson arrivals in geometric bulks:

\[ V_1(z) = \frac{(\sigma - \lambda/\alpha)(z-1)}{z[e^{\lambda(1-z)/(1-(1-\alpha)z)} + \sigma - 1] - \sigma}; \]

\[ \overline{N}_1 = \frac{\lambda(2-\lambda)}{2(\sigma\alpha - \lambda)} = \frac{\alpha(2/\alpha - \bar{a})}{2(\sigma - \bar{a})} \]  

(33)

where \( \bar{a} = \lambda/\alpha \) is the mean number of arrivals in a unit service interval.

(b) Geometric arrivals in geometric bulks:

\[ V_2(z) = \frac{(\sigma - \lambda/\alpha)[(1-\alpha)z-1]}{\lambda + \sigma(1-\alpha)z - \sigma}; \]

\[ \overline{N}_2 = \frac{\lambda}{\alpha(\sigma - \lambda)} = \frac{\bar{a}}{\alpha(\sigma - \bar{a})} \]  

(34)

where again \( \bar{a} = \lambda/\alpha \) denotes the mean arrival rate (per unit service interval).

In Figs. 4, 5 and 6 plots are given of the expected buffer occupancies \( \overline{N}_1 \) and \( \overline{N}_2 \) as functions of \( \bar{a} \) for different values of the mean bulk

![Fig. 4. The mean buffer occupancy \( \overline{N}_1 \) for a system with one output channel and bulk Poisson arrivals, versus the mean arrival rate; \( \sigma = 0.7 \) and \( \alpha = 0.1, 0.2, 0.4, 0.6, 0.8, 1 \) for curves a, b, c, d, e and f respectively.](image-url)
Example 2. Two output channels. In this case \( m = 2 \). The results of the analysis can still be given as explicit functions of the parameters of the system if geometric arrivals are assumed, but not in the case of Poisson arrivals. However, it is easily seen that \( A_2(z) \approx A_4(z) \) for \( z \) in the vicinity of the origin, i.e., the Poisson arrival process may be approximated by the geometric arrival process, if \( \lambda \ll 2 \). Since the maximum allowed value of \( \lambda \) in this case is \( m \alpha = 2 \alpha \), the approximation is good over a broad range of \( \lambda \)-values. We thus restrict ourselves to the case of geometric arrivals.

(a) The case where the two output channels are controlled by the same gate. From (25) and (26) we have the following expressions:

\[
V_4(z) = \frac{1 - z^*}{\alpha} \cdot \frac{1 - (1 - \alpha)z}{z - z^*},
\]

\[
\bar{N}_4 = \frac{1}{z^* - 1} \cdot \frac{1 - \alpha}{\alpha}
\]

where \( z^* \) can be expressed as explicit function of the parameters of the system as follows:

\[
z^* = \frac{\alpha \sigma}{2[\lambda + \sigma(1 - \alpha)]} \times \left( 1 + \sqrt{1 + \frac{4}{\alpha^2} \left[ \frac{\lambda}{\sigma} + (1 - \alpha) \right]} \right).
\]

In Fig. 7 plots of \( \bar{N}_4 \) versus \( \bar{a} = \lambda/\alpha \) are given, for various values of \( \alpha \), i.e., for various values of the mean bulk size in the arrival process.
(b) The case where each of the two output channels is controlled by its own gate. From (31) and (32) we have

\[ V_8(z) = \frac{1 - z^*}{\alpha} \cdot \frac{1 - (1 - \alpha)z}{z - z^*} \]

\[ \bar{N}_8 = \frac{1}{z^* - 1} - \frac{1 - \alpha}{\alpha} \]

where \( z^* \) is given by

\[ z^* = \frac{\sigma [2 - (2 - \alpha)\sigma]}{2[\lambda + (1 - \alpha)\sigma(2 - \sigma)]} \]

Figure 8 gives plots of \( \bar{N}_8 \) versus \( \bar{a} = \lambda/\alpha \) for various values of \( \alpha \), i.e., for various values of the bulk size in the arrival process.

It is clear from Figs. 7 and 8 that, as in the case \( m = 1 \), the performance (in terms of mean buffer occupancy) is better for low mean bulk sizes \( 1/\alpha \) than for high ones.

In Fig. 9 the four alternatives: single/bulk arrivals—one gate per output channel—one gate for the two channels, are compared. A value of 0.5 for \( \alpha \) and a parameter \( \alpha = 0.3 \) of the bulk size distribution are considered. The case where each of the output channels is controlled by its own gate (curves \( a', b' \)) clearly leads to a lower mean buffer occupancy than the case where there is only one gate for the two output channels (curves \( a, b \)), but the difference is rather small.

**Conclusions**

We have considered a discrete buffered system with infinite waiting room and periodic service opportunities, with a general (bulk) arrival process and a stochastic number of available output channels. The results found by Hsu [5] and Georganas [3] were found as special cases of the analysis and in that sense the analysis is a generalisation of their work. Several other examples were worked out in detail, showing among other things that a bulk arrival process generally leads to a higher mean buffer occupancy (and hence a higher mean
waiting time) than an input process with single arrivals, for the same value of the arrival rate.

Appendix A

\[ E(z) \triangleq z^m - z^m \cdot C\left(\frac{1}{z}\right) \cdot A(z) \]

has exactly \( m \) zeros inside the unit circle in the \( z \)-plane if \( \alpha < \alpha \).

We make use of Rouché’s Theorem [6]: If \( f(z) \) and \( g(z) \) are analytic functions of \( z \) inside and on a closed contour \( C \), and \( |g(z)| < |f(z)| \) on \( C \), then \( f(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( C \).

Let

\[ f(z) = z^m, \]
\[ g(z) = -z^m \cdot C\left(\frac{1}{z}\right) \cdot A(z), \]

then \( f(z) \) is analytic in the whole \( z \)-plane and \( g(z) \) is analytic inside the unit circle in the \( z \)-plane: \( \{ z : |z| \leq 1 \} \). Indeed

\[ g(z) = \left[ -z^m C\left(\frac{1}{z}\right) \cdot A(z) \right] = \left[ -z^m \sum_{i=0}^{m} c(i) z^{-i} \right] \cdot A(z) = \left[ -z^m \sum_{i=0}^{m} c(i) z^{-i} \right] \cdot A(z) \]

where the first factor is a polynomial in \( z \) and hence analytic in the whole \( z \)-plane and the second factor is analytic in the unit circle \( \{ z : |z| \leq 1 \} \).

Thus we can choose (Fig. 10):

\[ C = \{ z : |z| = 1 \}. \]

For \( z \in C \), set \( z = e^{i\theta}, 0 \leq \theta < 2\pi \). Then

\[ |f(z)| = |e^{i m \theta}| = 1 \]
\[ |g(z)| = |e^{i m \theta} \cdot C(e^{-i \theta})| \cdot |A(e^{i \theta})| \]
\[ = |C(e^{i \theta})| \cdot |A(e^{i \theta})| \]
\[ < 1 \]

for all \( \theta \neq 0 \).

Thus \( |g(z)| < |f(z)| \) for all \( z \neq 1 \).

Let us change the contour \( C \) so as to make an internal point of the point \( z = 1 \), which is obviously a zero of \( E(z) \). Change \( C \) into (Fig. 10):

\[ C' \triangleq C \cup \lim_{\epsilon \to 0} C. \]

where \( C \)

\[ C \triangleq \{ z = 1 + \epsilon e^{i \alpha}, -\frac{1}{2} \pi < \alpha < \frac{1}{2} \pi \} \]

is a semicircle with radius \( \epsilon \) outside \( C \).

For \( z \in C \): \( z = 1 + \epsilon e^{i \alpha} \)

\[ |g(z)| < |f(z)| \Rightarrow |C\left(\frac{1}{z}\right)|^2 \cdot |A(z)|^2 < 1. \]
\[ |A(z)|^2 = |A(1 + \epsilon e^{i \alpha})|^2 \]
\[ = \left| \sum a(n)(1 + \epsilon e^{i \alpha})^n \right|^2 \]
\[ = \left| \sum a(n)(1 + n \epsilon e^{i \alpha} + o(\epsilon)) \right|^2 \]
\[ = |1 + \epsilon e^{i \alpha} \alpha + o(\epsilon)|^2 \]
\[ = 1 + 2 \epsilon \alpha \cos \alpha + o(\epsilon). \]

Similarly

\[ |C\left(\frac{1}{z}\right)|^2 = 1 - 2 \epsilon \alpha \cos \alpha + o(\epsilon). \]

Hence, for \( z \in C \),

\[ |A(z)|^2 \cdot |C\left(\frac{1}{z}\right)|^2 = 1 + 2 \epsilon (\alpha - \varepsilon) \cos \alpha + o(\epsilon) \]

< 1

since \( -\frac{1}{2} \pi < \alpha < \frac{1}{2} \pi \) and \( \alpha < \varepsilon \).

Thus, for all \( z \in C' \) \( |g(z)| < |f(z)| \) and from Rouché’s Theorem it follows that \( E(z) \) and \( f(z) = z^m \) have the same number of zeros \( (m) \) inside \( C' \), i.e., \( E(z) \) has exactly \( m \) zeros in the unit circle \( \{ z : |z| \leq 1 \} \), one of which equals 1.

References