BUFFERS WITH STOCHASTIC OUTPUT INTERRUPTIONS

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A discrete-time buffer system with one randomly interrupted output channel is considered. It is shown how relationships can be found between the probability generating functions of the buffer occupancy (expressed in number of messages) at various time instants. The use of these relationships is illustrated.

Description of buffer system: We consider a buffer system with one or more input channels, an infinite waiting room and one single-output channel. Messages arrive at the buffer in a stochastic manner, wait in the buffer for some time and are then taken out via the output channel (Fig. 1). We assume that the

Fig. 1 Buffer system with random server interruptions

messages have a fixed length and that the output channel transmits data with constant speed. This implies that the service time of the messages is constant. The clock time period is defined as the time required to service one message. The data transmission is synchronous, i.e. the data is taken out synchronously from the buffer for transmission at each discrete clock time. However, this can only happen if the switch S is closed, i.e. if the output channel is available. Whenever the switch S is open, no transmission can take place.

The interruption process of the output channel is characterised by allowing the channel to be in one of two different states: A or B, i.e. available or blocked. The probability distributions of the lengths of 'available periods' and 'blocked periods', and the possible correlations between those lengths are irrelevant for the discussion and thus need not be specified.

The arrival process is described in terms of a sequence of IID random variables which denote the number of messages that arrive in the buffer during the consecutive clock time intervals. Let \( e(n) \) and \( E(z) \) denote the probability density function and the probability generating function, respectively, of these stochastic variables, i.e.

\[
e(n) = \text{Prob} \left[ n \text{ messages arrive during a clock time interval} \right]\]

and

\[
E(z) = \sum_{n=0}^{\infty} e(n)z^n
\]

Finally, let us introduce the random variables which denote the number of messages in the buffer a long time after the first arrivals, when a stochastic equilibrium has been reached, at various time instants:

\( c \) = buffer occupancy, as seen by an arrival
\( d \) = buffer occupancy, as left behind in the buffer by a departure
\( n \) = buffer occupancy at the beginning of a random clock time period
\( v \) = buffer occupancy at the beginning of a random clock time period during which the output channel is available.

Let \( C(z) \), \( D(z) \), \( N(z) \) and \( V(z) \) denote the probability generating functions of \( c, d, n \) and \( v \), respectively.

Relationships between \( C(z) \), \( D(z) \), \( N(z) \) and \( V(z) \):

(a) In any system where the state of the system changes by unit step values only (positive or negative) it is true that the equilibrium distribution of customers found by a new arrival is the same as the equilibrium distribution of customers left behind by a departure. Since we consider a single-output buffer system here, this result applies and thus we have

\[
C(z) = D(z)
\]

(b) A departure from the described buffer system can only occur at the end of a clock time period during which the server state was A. A departure actually happens if and only if the number of messages in the buffer is at least equal to 1 at the beginning of this clock time period. It follows that

\[
d = v + 1 + e, \quad \text{if} \quad v > 0
\]

where \( e \) denotes the number of arrivals during the clock time intervals which precedes the departure and where we must restrict ourselves to clock time intervals where \( v > 0 \). Since we have assumed that the arrivals during the consecutive clock time intervals are uncorrelated, this leads to

\[
D(z) = E[z^d] = E[z^{d-1}e | v > 0] = \frac{E(z)}{z} E[z^e | v > 0] = \frac{E(z)}{z} \frac{V(z) - V(0)}{1 - V(0)}
\]

(c) Consider an arbitrary arrival to the buffer system. Let \( I \) denote the clock time interval during which this arrival occurs and \( e_I \) the number of arrivals during \( I \). Then, owing to the fact that we consider an arbitrary arrival, we have

\[
\text{Prob} \left[ e_I = k \right] = \frac{kE(k)}{E[\text{Arr}]}\quad k = 1, 2, 3, \ldots
\]

where \( E[\text{Arr}] \) denotes the mean number of arrivals per clock time period. Indeed, the probability that \( I \) contains \( k \) arrivals is proportional to (i) the relative occurrence of clock time intervals with \( k \) arrivals and (ii) the number \( k \) itself, because our random arrival can be any of the \( k \) arrivals in a clock time period during which \( k \) arrivals occur.

Let \( f(n) \) denote the probability that there are \( n \) arrivals before our randomly chosen arrival, during \( I \). Then \( f(n) \) can be expressed as

\[
f(n) = \sum_{k=0}^{n} \text{Prob} \left[ e_I = k | g(n + 1 | k) \right] g(n + 1 | k), \quad n = 0, 1, 2, \ldots
\]

where \( g(n+1 | k) \) denotes the probability that our randomly chosen arrival is the \( (n+1) \)th arrival during \( I \), given \( I \) contains \( k \) arrivals, and is thus given by

\[
g(n+1 | k) = \frac{1}{k}
\]

It follows that \( f(n) \) is given by

\[
f(n) = \frac{1}{E[\text{Arr}]} \sum_{k=0}^{n} e(k)
\]

with corresponding probability generating function

\[
F(z) = \frac{E(z) - 1}{(z - 1)E[\text{Arr}]}
\]

Let \( n^* \) denote the number of messages in the buffer at the beginning of \( I \). Then we have the following relationship:

\[
c = n^* + f
\]
where $f$ denotes the number of arrivals during $l$ before our randomly chosen arrival, which has probability generating function $F(z)$. Owing to the mutual independence of the arrivals during the consecutive clock time periods, $f$ and $n^*$ are mutually independent and $n^*$ has the same probability distribution as $n$, the buffer occupancy at the beginning of a random clock time interval. It follows that

$$C(z) = N(z)F(z)$$

or

$$C(z) = \frac{E(z) - 1}{(z - 1)E[Arr]} N(z)$$  \hspace{1cm} (3)

Notice that eqn. 3 applies for every discrete buffer system where the arrivals are uncorrelated, irrespective of the number of output channels of the buffer. Eqn. 3 even applies if the messages have nonidentical lengths.

(d) For buffers with one single-output channel and fixed-length messages eqns. 1–3 can be combined, and a fourth relationship follows:

$$N(z) = \frac{E[Arr]}{1 - V(0)} \cdot \frac{(z - 1)E(z)}{z - E(z) - 1} \cdot V(z) - V(0)$$  \hspace{1cm} (4)

Applications: In Reference 4 a single-output discrete buffered system with random server interruptions and uncorrelated arrivals was analysed under the condition that the ‘available periods’ of the output channel have geometrically distributed lengths with parameter $\sigma$. As a result of the analysis the following expressions for $V(z)$ and $N(z)$ were found:

$$V(z) = \left(1 - \frac{E[Arr]}{\sigma}\right) \frac{(z - 1)E(z)[z + (1 - z)P_d E(z)]}{z - E(z)[z + (1 - z)P_d E(z)]}$$ \hspace{1cm} (5)

$$N(z) = \left(\frac{\sigma - E[Arr]}{\sigma}\right) \frac{(z - 1)E(z)}{1 - E(z)} \times \frac{1 - E(z)[z + (1 - z)P_d E(z)]}{z - E(z)[z + (1 - z)P_d E(z)]}$$ \hspace{1cm} (6)

Here $\sigma$ denotes the long-run probability of having the output channel available and $P_d(z)$ is the generating function of the blocked periods of the output channel.

It is clear that, using eqn. 4, eqn. 6 could have been derived directly from eqn. 5. Furthermore, using eqn. 3 we can find the distribution of the buffer occupancy as seen by the messages that arrive to the buffer, which is of interest when it comes to derive expressions for the distribution of the waiting time of the messages:

$$C(z) = \left(1 - \frac{E[Arr]}{\sigma}\right) \frac{E(z) - 1 - E(z)[z + (1 - z)P_d E(z)]}{z - E(z)[z + (1 - z)P_d E(z)]}$$  \hspace{1cm} (7)

In Reference 4 it was shown that eqns. 5 and 6 reduce to the results of Hsu, who characterises the interruption process by the single parameter $\sigma$, if we let

$$x = \sigma$$ \hspace{2cm} (8)

$$P_d(z) = \frac{\sigma z}{1 - (1 - \sigma)z}$$ \hspace{2cm} (9)

and

$$E(z) = \exp \{ - \lambda (1 - z)\}$$ \hspace{2cm} (10)

$$E[Arr] = \lambda$$ \hspace{2cm} (11)

for a Poisson arrival process.

Heines studies the same system as Hsu by considering the buffer occupancy at service completion times. The expression in eqn. 7 is the generalisation of Heines result to the more general interruption process we introduced in Reference 4. It is easily seen that eqn. 7 reduces to the expression found by Heines if we introduce the substitution eqns. 8–11. This also shows that Heines result could have been deduced directly from Hsu's result by simply using eqns. 2 or 3, without the need of a whole new method of analysis.

In References 5 and 6 Hsu's analysis was generalised to buffer systems with multiple-output channels, subject to random interruptions. As a result expressions were found for the probability generating function of the buffer contents at random clock times. It is clear that the corresponding expressions for the buffer contents, as seen by the arrivals, can be obtained by applying the eqn. 7.

In general, if we are able to determine an expression for $V(z)$ for whatever discrete-time buffer system with one single server and uncorrelated arrivals, eqns. 1–4 give us a means to determine $C(z)$, $D(z)$ and $N(z)$ immediately, without any further complications.

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