Comments on "Discrete-Time Queueing Systems and Their Networks"

HERWIG BRUENEEL

Abstract—In the paper, Bharath-Kumar presented an analysis of a discrete buffered system of infinite capacity, where the length of the packets is drawn from a geometric distribution and the packets leaving the resource have to be retransmitted with some probability. In this correspondence an analysis of the same system is given which applies for a general packet length distribution, and where the length of a packet remains the same each time it is retransmitted, as opposed to Bharath-Kumar’s model where the retransmission packet lengths are independently chosen.

A discrete buffered system is considered to which packets arrive during each time slot in i.i.d. bulks. Let \( P_A(z) \) be the probability generating function of the number of arriving packets during each time slot. The packets have i.i.d. packet lengths and \( P_A(z) \) denotes the probability generating function of these lengths. Packets which have received service either leave the system with probability \( 1 - \gamma \) or have to be retransmitted with probability \( \gamma \). Hence, the probability that a packet must be transmitted \( n \) times is given by \( (1 - \gamma)^{n-1} \gamma^n \).

In the paper, a rather elegant analysis of the system, resulting in the probability generating function of the buffer contents at arbitrary time slot marks, was possible due to 1) the assumption of geometrically distributed packet lengths and 2) the assumption that the packet lengths of the retransmitted packets are independently chosen. We will present here a more general analysis by considering the system at customer departure times. As a result, we will find the probability generating function of the buffer contents at service completion times. From this function we will also derive the probability generating function of the number in system at arbitrary slot marks. Let \( u(k) \) be the number of packets in the buffer just after the \( k \)th packet leaves the buffer. Then the following equations are of interest:

\[
\begin{align*}
  u(k + 1) &= u(k) - 1 + v(k + 1) \quad \text{if } u(k) \neq 0; \\
  u(k + 1) &= w(k + 1) \quad \text{if } u(k) = 0.
\end{align*}
\]

Here \( v(k + 1) \) denotes the number of packets arriving between the departures of the \( k \)th and the \((k + 1)\)th packets, i.e., during the time needed to service the \((k + 1)\)th packet, and \( w(k + 1) \) denotes the number of packets that arrive in the buffer during the time the \((k + 1)\)th packet remains in the system, i.e., \( w(k + 1) \) includes 1) the arrivals during the time needed to service the \((k + 1)\)th packet and 2) the packets which arrive in the same time slot as the \((k + 1)\)th packet, after this packet.

Let \( N \) indicate the number of packets in the system at service completion times when the system has reached an equilibrium and \( P_N(z) \) its probability generating function. Heimes [1] showed that for the system described by (1) and (2), the following result holds:

\[
P_N(z) = \frac{1 - \frac{dV}{dz}}{1 - \frac{dV}{dz} + \frac{dW}{dz}} \cdot \frac{zW(z) - V(z)}{z - V(z)}.
\]

Here \( V(z) \) and \( W(z) \) denote the probability generating functions of \( v(k + 1) \) and \( w(k + 1) \), respectively (independent of \( k \)).

Let \( s \) indicate the time needed to service one packet, expressed in time slots. It is easily shown that

\[
V(z) = s(P_A(z)).
\]
and

$$W(z) = \frac{P_A(z) - P_A(0)}{z[1 - P_A(0)]} \cdot S(P_A(z))$$

where $S(z)$ is the probability generating function of the random variable $s$. Hence,

$$P_N(z) = \left( \frac{1}{E[A]} - E[s]\right) \cdot \left(P_A(z) - 1\right) \cdot \frac{S(P_A(z))}{z - S(P_A(z))}$$

(3)

where $E[A]$ is the mean arrival rate.

The function $S(\cdot)$ and the quantity $E[s]$ can be determined as follows. Let $L$ denote the length of an arbitrary packet and $T$ the number of times this packet has to be transmitted before leaving the buffer. Then $s = L \cdot T$ and

$$S(z) = E[z^s] = E_T[E[z^{L \cdot T} | T]] = E_T[P_L(z^T)]$$

$$= \sum_{n=1}^{\infty} (1 - \gamma)^{n-1} P_L(z^n).$$

(4)

Furthermore,

$$E[s] = E[L] \cdot E[T] = \frac{E[L]}{1 - \gamma}.$$

(5)

Substitution of (4) and (5) in (3) yields an expression of $P_N(z)$ in terms of known quantities only.

We now derive the probability generating function $P_{Narb}(z)$ of the random variable $N_{arb}$ which denotes the number of packets in the buffer at arbitrary slot marks. First, we notice that the number in the buffer found by an arrival—which will be indicated by $N'$—is identical in distribution to the number left behind by a departure, i.e., the quantity $N$ defined above [2]:

$$\text{Prob} \left[ N' = n \right] = \text{Prob} \left[ N = n \right], \quad n = 0, 1, 2, \ldots $$

(6)

Now it is clear that the number in system $N'$ seen by an arbitrary arrival can be written as

$$N' = N^* + R$$

(7)

where $N^*$ denotes the number in system at the slot mark just prior to the arrival and $R$ the number of packets that arrived before the arrival of interest, during the same time slot. It is easily seen that, due to the memorylessness of the arrivals from slot to slot, $N^*$ and $N_{arb}$ are identical in distribution, i.e.,

$$\text{Prob} \left[ N^* = n \right] = \text{Prob} \left[ N_{arb} = n \right], \quad n = 0, 1, 2, \ldots $$

(8)

From (6)-(8) it follows that

$$P_N(z) = P_R(z) \cdot P_{Narb}(z)$$

(9)

where $P_R(z)$ is the probability generating function of the random variable $R$, i.e.,

$$P_R(z) = \sum_{k=0}^{\infty} z^k \text{Prob} \left[ R = k \right].$$

Here $\text{Prob} \left[ R = k \right]$ can be written as

$$\text{Prob} \left[ R = k \right] = \sum_{l=k+1}^{\infty} f(l) \cdot q(k+1 | l)$$

where $f(l)$ denotes the probability that there are $l$ arrivals in the time slot during which the considered arrival enters and $q(k+1 | l)$ denotes the probability that this arrival is the $(k+1)$th of the interval, given there are $l$ arrivals in the interval. Since the arrival under consideration was chosen at random we have

$$q(k+1 | l) = 1/l.$$ 

Furthermore, the probability that the time slot, during which the arrival comes, contains $l$ arrivals is proportional to the bulk size $l$ and to the relative occurrence of this bulk size, and thus $f(l)$ is given by

$$f(l) = \frac{l \cdot p_A(l)}{E[A]}$$

where $p_A(l)$ is the probability of having $l$ arrivals in a time slot.

It follows that

$$\text{Prob} \left[ R = k \right] = \sum_{l=k+1}^{\infty} \frac{p_A(l)}{E[A]}$$

and

$$P_R(z) = \frac{1}{E[A]} \sum_{k=0}^{\infty} z^k \sum_{l=k+1}^{\infty} p_A(l)$$

$$= \frac{1}{E[A]} \cdot \frac{P_A(z) - 1}{z - 1}.$$  

(10)

From (3), (9), and (10) we find the expression for the probability generating function of the number of packets in the buffer at arbitrary slot marks:

$$P_{Narb}(z) = (1 - E[A] \cdot E[s]) \cdot (z - 1) \cdot \frac{S(P_A(z))}{z - S(P_A(z))}$$

(11)

For the special case where the packet length is geometrically distributed, i.e., where

$$\text{Prob} \left[ L = n \right] = \sigma(1 - \sigma)^{n-1}, \quad n = 1, 2, \ldots$$

or

$$P_L(z) = \frac{\sigma z}{1 - (1 - \sigma)z}$$

our results do not reduce to Bharath-Kumar’s because of the different assumptions concerning the length of the retransmitted packets. For the case of independently chosen retransmission packet lengths, as in Bharath-Kumar’s model, we would have

$$s = \sum_{l=1}^{T} L_l$$
where the $L_i$'s are i.i.d. with probability generating function $P_L(z)$, and thus

$$S(z) = \frac{(1-\gamma)P_L(z)}{1-\gamma P_L(z)}.$$ 

It is easily shown that if we use this expression for $S(z)$ instead of the one in (4), our model yields the same results as Bharath-Kumar's for geometrically distributed packet lengths.

REFERENCES
